Short, but yet Plain

ELEMENTS

OF

GEOMETRY

AND PLAIN

TRIGOMETRY.

Shewing how by a Brief and Easie Method, all that is Necessary and Useful in *Euclide*, *Archimedes*, *Apollonius* and other Excellent *Geometricians*, both Ancient and Modern, may be Understood.

Written in *French*

By *F. Ignatius Gaston Pardies.* K

And now rendred into *English* from the Fourth and Last Edition.

By *John Harris* M. A. and F. R. S.

With many Additions, and Improvements: The whole being Accommodated to the Capacities of Young Beginners.

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TO

My Worthy Friend
Charles Cox Esquire,

Member of Parliament for the Burgh of *Southwark.*

Dear SIR,

A*Mong the many Obligations you have conferr'd upon me, I account it not the least, that you have given me a* Rise *to revive my* Mathematical Studies; *in which as I have formerly spent some time, so I know of no more* Useful way *of employing my leasure Hours.*

And indeed Sir, the Diversion *and* Advantage *I have lately reaped from them hath (by the Divine Blessing) both supported me under, and in a good Measure carried me through such Pressures and Difficulties as I once almost Despaired of Surmounting.*

The Mathematick Lecture *which you have set up* Gratis *in your* Burgh, *is a Demonstrative Proof both of your sincere Endeavour to promote the Good of your Country, and also of your Capacity to do it the best way. And I hope to see such Effects from so Generous a Design, as will render your Name justly Honourable to Posterity as well as to this present Age. Sir, you know* your Self *and* Me *too well to take this for Flattery. 'Tis what* Truth, Justice *and* Gratitude *Oblige me to say.*

I shall only add, that I am glad of this opportunity to shew the Just Esteem I have of your Merit, and the equal regard I have for your Friendship. I am

SIR,

your most Obliged

Humble Servant,

John Harris.

THE

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Author's Preface.

THose who ſhall compare the ſmall Bulk of this Book with the bigneſs of its Title, may perhaps at firſt be diſcouraged from Reading it, by the diſproportion that appears between the one and the other. And they may be enclined to fear, that ſuch extraordinary Promiſes and Pretences are to be taken for the too Confident Expreſſions of one who hath raſhly undertaken what he knows not how to perform. But I beg of them a little to ſuſpend their Judgments, and then they will find that I do not here Publiſh much above half *The Elements*; and that of the XVI. Books which the whole ought to contain, I have now ſet forth but IX. The Reaſon of which is, that the others Explaining what is moſt Profound and Sublime in the extraordinary Inventions and Improvements of *Geometry*, are not ſo neceſſary for thoſe who would begin to learn that Science. Nevertheleſs in theſe firſt Books, we have not omitted any thing that is Good and Uſeful throughout the 15 Books of *Euclid*; and we have added beſides what *Archimedes* hath Demonſtrated of the

Quadrature of the Circle; the Squaring of *Hippocrates* his Lunes, the Doctrine of Logarithms, Sines, Tangents, Secants, *&c.* and many other things of that Nature. You may ſee here thoſe wonderful Properties of Numbers which *Euclide* hath Demonſtrated in the *Seventh*, *Eighth* and *Ninth* Books of his Elements. You may learn here the Demonſtration of *Incommenſurable Quantities*, which perhaps is the greateſt Effort, the Mind of Man is capable of; ſince by ſearching into the Poſſibility of Things, it diſcovers with the greateſt Clearneſs, what is, and what is not. And that among the infinite Multitude of Compariſons (or *Ratios*) that there are, all which it ſees Poſſible to be between two Magnitudes, it Demonſtrates with indubitable Certainty, that even God himſelf doth not ſee any one of them, capable to be a Common Meaſure to thoſe two Magnitudes. But while ſuch kind of Demonſtrations are fine, they muſt be owned to be very Difficult.

Thoſe to whom we are Obliged for ſo great a Diſcovery, have not ſhewed us any other way, than what they went in themſelves, whether in Reality they knew no other way; or whether they were deſirous we ſhould try part of their Pains; and ſo gain a more delightful Taſte and Reliſh of the Pleaſures of that New World which we have now Diſcovered with greater Labour our ſelves. Be that as it will: Their way is ſo long and difficult, that very few have

have been Maſters of ſo much Patience and Conſtancy, as to bear the Tedionſneſs of it, or ſo much ſtrength as to get over the Fatigue. I know not whether I may dare ſay, that I have been ſo happy as to diſcover a ſhorter way. That will be no very great Honour for me: A daring ordinary Sailor is ſometimes more Lucky in making new Diſcoveries, than the moſt Skillful and Wiſe Pilot. And bare chance may help him to find, by a Tempeſt, that which he would never have known how to have Diſcovered by all his Knowledge of the Sea. It may be alſo, that running, as I have done, over thoſe vaſt Oceans of *Geometry*, Fortune might guide me into ſome new Path, unknown to thoſe great Men that have gone before me. I don't pretend nevertheleſs to Attribute ſuch good Luck to my ſelf: But this I may ſay, at leaſt, that the way which I take to come to the Knowledge of *Incommenſurables*, is very ſhort and eaſie; and that by the ſmall Attention which is neceſſary only to Read over four or five ſmall Pages, ſomething may be Comprehended, which very few Perſons, even of thoſe which have look'd into *Geometry*, are capable to underſtand without it.

After this, I Treat of the ſeveral Kinds of Progreſſions, and inſiſt more particularly on thoſe two Celebrated ones *Arithmetical* and *Geometrical*; which comparing together, I Diſcourſe of Logarithms, and Diſcover an Artifice, by the help of a Geometric Line,

which

which will be very useful in Algebra, for Resolving Problems of all Dimensions. This is that Line by which formerly I Squared the Hyperbola; and that which one of my Friends shewed me a little while since, which hath been Published in the *English Philosophical Transactions* about this Matter by some very learned Geometricians, did not at all surprise me; but rather made me think, that those Gentlemen were not willing to Communicate to us, all they could say on that Subject. I finish this *First* Part, with the Practice of Geometry; which ought to have been the last Book of all the Elements. Where, besides those *Problems* and *Practices* that are most easie and common, I have given also the Principles of *Measuring of all Magnitudes*; of taking the distance of *Inaccessible Places*, and of drawing the Plane and making a Map of any Place or Country; of finding the Sines, Tangents and Secants of all Angles; and in fine of obtaining the Knowledge of all that, which is called *Practical Geometry*.

After this I shall give you, in so many several Books, *Algebra*, *Conick-Sections*, *Sphericks* and *Staticks*. But above all I Establish five or six General Rules, by which afterwards as by so many Corollaries, you may Demonstrate an infinite Number of Propositions, which now pass for very great things in Geometry. By these may be discovered the Nature and the Measures of those *Asymptotique* Spaces, the

Knowledge of which is the moſt wonderful thing in the World; and which ſhews very clearly both the Mighty Power and the Spiritual Nature of our Souls; becauſe by the ſole Light of the Mind, penetrating beyond Infinitely, it diſcovers ſo clearly thoſe things, which no ſenſible Experience can bring to our Knowledge, nor any Corporeal Power alone enable us to perceive. Theſe *Spaces* are an Extenſion actually Infinite, Comprehended between two Lines, which tho' Infinitely Produced, yet will never meet (*though, as he ſhould have ſaid, they continually approach nearer to each other*). From whence they derive their Name of *Aſymptotes.* Notwithſtanding, we Demonſtrate that theſe Spaces, tho' Infinite in Length, are nevertheleſs equal to a Circle or to ſome other determinate Figure. So that even an Infinite thing, tho' Immenſe and Innumerable, is brought within the Calculation and Meaſure of Geometry, and our Mind, which is yet greater and more Capacious than it, is able to Comprehend this.

Of all Natural Knowledge that a Man can acquire by his own proper Reaſoning, doubtleſs the moſt admirable is this Comprehenſion of Infinite: And I don't ſee any thing more proper to Convince us of the Exiſtence of our (*Immaterial*) Souls; and to make us conſider, that beſides that material Faculty which we have to Perceive or Imagine things, by the help of the Organs of our Body; we

have

have ſomething in us that is entirely Spiritual, which can Think, and which can Reaſon, and which the greateſt of all Philoſophers calls, *A Power Independent upon the Organs, ſeparate from Matter, and by no means deriving its Original from our Body.* In Reality, whatever Effort we make to Imagine *Infinite*, we ſhall never be able to compaſs it: And as long as we keep to Imagination only, we may gain an Idea of a Space vaſtly Extended, but it will be always Limited: Becauſe the Imagination being, properly ſpeaking, a Corporeal Power, which repreſents nothing to us but by Fancy, and by ſenſible Images, it muſt needs be its ſelf, even as a Body is, Limited in its Repreſentations. And as no Draught can repreſent to our Eyes a Space actually Infinite, becauſe that which is bounded within a certain Space cannot contain that which hath no Bounds or Limits; ſo the Imagination being nothing but a Draught or Picture that repreſents to us Images, indeed very Subtile, but always Material, it can ſhew us nothing but Corporeal and Limited things; whatever is Immenſe and Infinite, not being Poſſible to be contained within the bounds of a Corporeal Repreſentation. Imagination therefore can by no means reach up to that Power which Repreſents to us Infinity. But then on the other hand, that Demonſtration which we have of the Nature and Properties of this Immenſe and Infinitely extended Aſymptote, convinces us equally,

that

that we have in us a Faculty capable of Representing to us this Infinite Extension. For as in Order to Measure, by Scale and Compass, any Figure drawn upon Paper, that Figure must be Represented to my Eyes, and to be come at by my Hand, so that applying my Instrument to its Angles and Sides, I may take all its Dimensions, and thence Determine its Magnitude: So also, that by the Rule of my Reason I may take the Measures of this Asymptote Space, 'tis necessary that I should have an exact Idea of it in my Mind: And that this same Mind applying it self, as I may say, to this Idea and to this Internal Figure, it may take its Dimensions, determine its Magnitude, and Demonstrate its Properties. We ought then to know, that we have in us clear and distinct Ideas of Infinite Extension: And that consequently, that Faculty which can represent it so to us, as nothing that is Corporeal can do, must be a Power purely Spiritual and distinct from Matter: So that Geometry by one single Demonstration, proves both the wonderful Properties of Nature, and the more Important Truths of Morality.

Shall I venture to go yet a good deal farther? And to say that in this same Demonstration we find also an Invincible Proof of the Existence of the Deity? I know that the Divine Nature is an Abysse of Light, diffusing himself every where; and that he makes himself known even to the Spirits of the most Blind and Stupid;

pid; But I know alſo to what a Pitch the Impiety of our Libertines is arriv'd: Who not being able to reſiſt the Convictions of their own Minds, nor to anſwer themſelves: They endeavour to Elude, outwardly, the Demonſtrations that others bring againſt them, by entrenching themſelves within the difficulties of Eternity; and they think to be very ſecure and to lye cloſe under the Infinite Multitude of dependent Cauſes, and to find always a Place of Refuge in the Eternal Chain or Succeſſion of Events and Productions.

But Geometry by the manifeſt Example of the Aſymptotes, demonſtrates Invincibly, that even in that pretended Chain or Succeſſion of Cauſes Subordinate and Dependent one upon another Infinitely, you muſt neceſſarily come to ſome firſt (*Cauſe* or Nature): Which becauſe Concurring with all thoſe particular Cauſes, and Correſponding to all Times, is alſo it ſelf Infinite and Eternal, and tho' not producing any of theſe Cauſes without the Concurrence and Determination of others; yet is nevertheleſs the General Cauſe which produces and conſerves all things.

Perhaps, after all, it will be thought, that I have put things here only by way of Abridgment; and that this Geometry may help the Memories of thoſe who have already learnt the Science, but will not ſerve to inſtruct thoſe who have a mind to learn it. I declare that this is very far from my Deſign, which never

was

was to make any Abridgment. I always intended to compoſe a Treatiſe of Geometry for young Beginners; by the help of which, even thoſe which had never heard of Mathematicks might underſtand in a little Time, not only that which is moſt neceſſary in Geometry, but even that alſo which is more Abſtruſe and Sublime. I know that on this Subject, the ſhorteſt Books are not always the cleareſt; and among thoſe many which pretend to render the underſtanding of *Euclide* eaſie to us, many have leſſen'd the Volume; but all have not by that means much ſhortned the Time one muſt ſpend to Underſtand him. Amongſt all the Commentators on that Book, the longeſt, I think, is *Clavius*, and Father *Fournier* is the ſhorteſt: Yet I am perſwaded more time muſt be employ'd to Underſtand *Fourniers Euclide*, then *Clavius*: So true is it in Geometry, that we ought not to Meaſure the Time of Learning the Science by the ſmallneſs or greatneſs of the Volume. Thus in this Deſign, which I Publiſh as a means to render the Knowledge of this Science as eaſie as is poſſible, I have not ſo much Studied to be ſhort in what I have written, as to render my ſelf Intelligible in the method of my Proceeding: And if this Book appear very ſmall, it is ſo, not from the Brevity of particular Demonſtrations, but from the Facility of the general Method.

For

For it ought to be obſerved that one of thoſe things which render the Reading of *Euclide*, and other common Authors, ſo very difficult and tedious, is, that by that rigorous Exactneſs which won't permit them to paſs any thing by Un-demonſtrated, how eaſie ſoever it appears otherwiſe, it often happens that what would have been Clear and Plain, if barely conſider'd by the Mind, and the truth of which would Naturally and at firſt Sight appear, becomes very Difficult, Tedious and Obſcure, when needleſly reduced to a regular Demonſtration. Moreover we find often times, that *Euclide*, for the Demonſtrating one important Propoſition, ſhall make uſe of a long Chain of others, which have no other Uſe but to Demonſtrate that Principal one.

If then by a bare Expoſition, the Truth may be made appear without giving one ſelf the trouble of Demonſtrating that, of which we are ſatisfied; and without Spinning out Diſcourſes which ſeem to be of no other Uſe but to make us miſapprehend that, which we could not be Ignorant of without them; A great deal of Pains may be ſaved. Alſo, if we can all at once Demonſtrate thoſe Important, Capital Propoſitions of *Euclide*, without ſuch a long Chain of preparatory Demonſtrations, we ſhall doubtleſs by that means Retrench many Uſeleſs things. And this is what I think I have done in many Places; Demonſtrating by one, what ordinarily uſed to be

done

done by a tedious Chain of many Propoſitions. Another means of Abridgment I have uſed; which is reducing things under certain General (*Heads* and) Principles; which I have done, not only in this Book, where by five or ſix General Rules I Demonſtrate an infinite Number of Propoſitions; but alſo in many other Places: As when treating of Conick-Sections, I Demonſtrate the Properties of *Four*, by any one Property peculiar to one ſingle Section. *v. gr.* Conſidering all under the Properties of the Ellipſis; I ſay that a *Circle* is an Ellipſe whoſe two *Focus* Coincide: That a *Parabola* is an Ellipſe whoſe *Focus* are infinitely diſtant from each other: And that the *Hyperbola* is an Ellipſe, whoſe *Focus* are more than infinitely diſtant. And this laſt is good Senſe as I explain it in that place.

Some one doubtleſs may think it amiſs, that I have omitted the common Method of Ranging the Definitions, Principles and Propoſitions; and he may believe perhaps that I have wrong'd Geometry by it, In taking any thing from a Science, which hath always been counted moſt Perfect and Exact. Another may Reproach me for retaining ſome *Old Faſhion* Demonſtrations, after that the *Moderns*, by a Politeneſs ſo proper to our Age, have given us Demonſtrations much *more Natural*, and ſhewed us the difference between *Enlightning the Mind*, and *Convincing it*. I may be told alſo, that I have been negligent in many things,

That I have advanced many Propoſitions without Demonſtrating them, that I often cite Places, which do not directly prove the Point in Queſtion, and that I make uſe of the *Converſe*, and of the *ſame Propoſition*, indifferently. To all which I anſwer in one word, that in the deſign which I had to Teach Geometry with all poſſible Eaſineſs, the way which I have gone, ſeem'd to me moſt proper: But neverthelesſs that ſhall not hinder me from profiting by the Advice of thoſe intelligent Perſons who will be ſo kind as to give it me.

But I find, that while I profeſs to be ſhort in the Book, I have been too long in the Preface: Therefore I ſhall not ſtay now to ſhew the great Advantages of the Knowledge of Geometry: Only this I ſay, that if ever there be any Advantage or Profit to be had in the Study of Natural Learning, or in the Practice of Arts; Geometry is abſolutely neceſſary to gain both. 'Tis well known to what an Height the perfection of Arts is arrived in our Age, and with what Penetration the moſt hidden and profound Matters of Phyſicks have been Dived into and Diſcovered. Inſomuch that 'tis found now a days, that Geometry is as neceſſary every whit as Mechanicks; which are nothing but Geometry applied to Local Motion. And thoſe Writers whoſe Books are now moſt in Vogue, are unintelligible to one that hath not *both* theſe Sciences. As for Mechanicks, I have given it you in one Part of the

Elements,

Elements, under the Diſcourſe on Local Motion, Which I ought not to be aſhamed to own for mine: And I hope that with what I have now publiſh'd in this Book of Geometry, there will be two good helps towards Underſtanding the Phyſicks of this Age, and to judge thoroughly of it. And perhaps it may be found that thoſe who have had the Reputation to have built their Philoſophy on the foundations of Geometry and Mechanicks, are not always Unſhakeable: And that the ſame thing which hath ſerved to make their Learning valuable, may help to diſcover their Errors.

I would Advertiſe my Reader alſo, that I by no means pretend to be the Author of that which I have given him in this Book: I have taken from all Parts that which was agreeable to my Deſign; and if any one find here any thing that he thinks is his own Invention, or any others; and will boldly take it away from me, and reſtore it to its proper Author, I freely conſent, and will by no means conteſt the Matter. But if by chance there ſhould be any thing met with, that is not to be found elſewhere, and that he is willing to Attribute it to me, I will then acknowledge it for mine, for fear it ſhould belong to No-Body.

THE TRANSLATOR TO THE READER.

After so long a Preface by our Author, I shall only tell you, that I judge this little Book to be the plainest, shortest, yet easiest Geometry I have yet seen Published: And therefore I thought it very well worth my while to let it appear in our own Language, as it hath already done twice in the Latin *Tongue. And 'tis so well esteemed of, by very Competent Judges amongst Us, as to be read in our Universities, by Tutors to their Pupils. As to the Translation; I have by no means Obliged my self servilely to follow the* French *way of Expression; for indeed a Literal Version of a Book out of any Language will scarce be Intelligible in* English. *I have therefore all along aimed rather to give you*

you F. Pardies *Senſe, than his Words; and have made him ſpeak what I judge he would have done, had he wrote in our Language. I have made no ſcruple, to add any thing that I ſaw neceſſary to render him clear and intelligible; though when I have enlarged much, you have it in the* Italick *Letter. Wherever I found a Demonſtration which wanted (as I thought) a better Method or a clearer Turn, I have endeavoured to give it one. And I hope it will fully Anſwer the Author's Deſign in Writing it and mine in Tranſlating it,* viz. *to render this moſt Noble and moſt Uſeful Science, eaſily attainable by young Beginners.*

Advice

Advice to thoſe who would Underſtand Geometry.

1. THey ought to enure themſelves to conſider well the *Figures*, at the ſame time as they Read the *Propoſitions*. There will be ſome Labour and Difficulty at firſt, but they will break thro' it in two or three Days.

2. They ought not to be diſcouraged, if they meet with ſome things which they do not underſtand at firſt; Geometry is not ſo eaſily to be attain'd, as Hiſtory.

3. If after they have Read and Conſider'd attentively any Propoſition, they find they don't Underſtand it; let it be paſſed over, it will probably be Intelligible by reading farther, or at leaſt when they have gone over the whole, and have began to Read it over anew. There are indeed many things in Geometry, that will never be well underſtood at firſt Reading over.

4. The Numbers which are within the Parentheſes *v. gr.* (3. 14.) ſhew that the Matter there ſpoken of hath been proved elſewhere, *viz.* in this Inſtance, in the fourteenth Article of the *Third* Book; And they ought always to mind the Number of the Article, and to conſult the Places referr'd to, that ſo they may

may gain the Demonſtration of what they Read.

5. When they meet with any Words which they dont Underſtand, they muſt conſult the Table at the End of the Book.

6. 'Tis good to have a Maſter at firſt, to Explain to them the Nature and Manner of the Demonſtrations: For by that means they will Underſtand the thing both much eaſier and much ſooner, than they can do by Reading by themſelves.

I hoped ſpeedily to have been able to publiſh the reſt of this Geometry, but I have been forced to defer the Impreſſion for ſome time, that I might publiſh the other Mathematical Treatiſes which are much more neceſſary. But as ſoon as I have finiſh'd my Staticks, Opticks and the Quadrants, which I am now upon, I will then Print the whole Courſe of Algebra, Conick-Sections and all the reſt which I have Promiſed, to compleat Geometry.

E L E-

ELEMENTS OF GEOMETRY.

BOOK. I.

Of Lines *and* Angles.

1. BY the Word *Quantity*, we mean that which being compared with another thing of the ſame Nature, may be ſaid to be Greater or Leſs than, Equal or Unequal to it. As Extenſion, Number, Weight, Time, Motion. And all thoſe things which are capable of being ſo compared as to More or Leſs, are the Object of Geometry.

2. We deſign neverthеleſs to conſider now only Extenſion; as being that which ſerves for an *Example* and *Rule* to Meaſure all other Quantities by.

3. That Quantity which being ſuppoſed without any Breadth or Thickneſs, is extended only in Length, is called *A Line*. That which hath both Length and Breadth, (but is ſuppoſed to have no Thickneſs) is called a *Surface* or *Superficies*: And that which hath Length, Breadth and Thickneſs, is called a *Body* or a *Solid*.

4. *A Point* is that which is considered as having no manner of Dimensions; and as being Indivisible in every Respect. The Ends or Extremities of Lines, as also the middle of them, are Points.

5. There are *Strait Lines*, and there are *Crooked* or *Curved* Ones: Also there are Even and Plain Surfaces which are called *Plan's*; and there are *Crooked* or *Curved* Ones: Which like a Vault, (*or the Tilt of a Boat or Waggon*) are *Convex* above, and *Concave* below.

6. When two Lines meet in a Point, The *Aperture*, *Distance* or *Inclination* between them is called an *Angle*. Which when the Lines forming it are Right or *Strait Ones*, is called a *Rectilineal Angle*, as A. But when they are *Crooked*, 'tis called a *Curvilineal One*, as B. And when one is *Strait* and the other *Crooked*, 'tis called a *Mixt Angle*, as C.

N. B. *The Lines forming any Angle are Called its Legs.*

7. That *Angle* is said to be *less* than another, whose Legs are more inclined to, (*or nearer to*) each other. Let there be Two Lines A B and A C meeting in the Point A. If you Imagine those Lines to be moveable like the Legs of a pair of compasses and yet fastened together on A, as with a Joint; 'Tis easie then to conceive, that the farther they are Opened or Parted from one another, the *greater* will be the Angle between them: As on the Contrary, the nearer they are brought together, the more will they incline towards each other, and so the Angle bewteen them must be the *less*.

8. If

8. It must therefore be observed that the *Quantity of Angles*, is by no means measur'd by the *Length* of their Legs, but by their *Declination*. Thus, *v.gr.* the Angle B is bigger than A; tho' the Legs of the Angle B are much shorter than those of A: But then those of A are much more inclined to each other, than those of B. And to apprehend this the Better, Imagine the Angle, B put upon A; as you may conceive by the Prickt Lines about A, which represent the Legs of B lying on it. For 'tis plain the Angle A will be easily contained within B; and that its Legs are much more inclin'd to one another, than those of B, and therefore it is less than B.

9. An Angle is usually marked by 3 Letters, of which the middlemost, and which always is placed in the Angular Point where the Lines meet, denotes the Angle. As in the Figure following *b a c* denotes the Angle made by the two Lines *b a* and *c a* meeting in the Point *a*.

10. If we Imagine the Line *a b* fastned by its End *a* in the middle of the Line *d c*, but yet so as to be moveable on *a* as on a Centre: If then you conceive it be moved quite round till it arrive at the place where it began, the Point *b* will describe a Curve Line, which is usually called a Circle; but 'tis rather the *Circumference of a Circle*; for properly speaking, the Circle is the *Space* contain'd within that Circumference.

11. Any part of the Circumference is called an Ark, as *b c*.

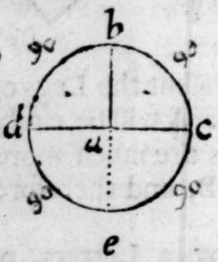
12. The Line *d c* (*passing through the Centre*) and terminated by the Circumference is called the *Diameter*, and divides the Circle into two equal Parts. Also every right Line passing thro' the Centre *a* (*and Terminated at each end by the Circumference*) divides the Circle into two equal Parts, and will be a Diameter.

13. The Line *a b* or *a c* or any other drawn from the Center to the Circumference is called the Radius or Semidiameter.

14. All *Radius*'s or *Semidiameters* (*of the same or equal Circles*) are equal (*as is plain from the Genesis of a Circle given in Art.* 10.)

15. When the End *b* of the Radius *a b* is equally distant from the two Ends of the Diameter *d c*; That is, when the Point *b* is in the very middle of the Semicircumference *d b c*; then will *b a* make two Angles with *d c* that are called *Right ones*: Which are equal one to another, that is, the Angle *d a b* is equal to *b a c*. And if the Line *b a* be produced below to *e* it shall then (with *d c*) make 4 Right Angles; and it will be another Diameter; which with the former *d c* will Divide the Circle into 4 equal Parts.

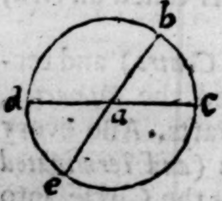
90×4=360



16. Then those Lines are said to be *Perpendicular* one to another: *viz.* *b a* to *d c*, and *d a* to *b e*.

17. But if *b* be nearer to one End of the Diameter (or Right Line) *d c* than it is to the other, it is then said to fall on the other *Obliquely*; and it makes with *d c* two Angles, that are *Unequal*: Of which the Lesser *b a c* is called *Acute*, and the Greater *d a b* is called *Obtuse*.

If the Line *a b* be produced to *e* it will be a new Diameter, and will make below two other Angles: So that in the whole there will be four Angles; of which those two that touch only in the Angular Point, as *b a c* and *e a d*, as also *d a b* and *e a c* are called *Vertical* or *Opposite* Angles. But those that have one Leg common to both, as *d a b* and *b a c*; and *b a c* and *e a c* are called *Adjoining or Contiguous Angles.*



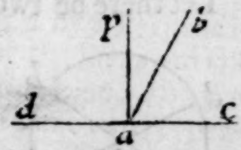
18. Those

18. Those Angles which (at Equal distances from the Angular Point) are subtended by Equal Arks are also equal themselves. As if the Ark *b c* be proved Equal to the Ark *d e*, then will the Angle *b a c* be Equal to *d a e*.

19. The two *Contiguous Angles* taken together, are always equal to two Right ones.

For as the Line *d c* is a Diameter, and therefore cuts the Circle into two Equal Parts, the two Arks *d b* and *b c* taken together, will be equal to a Semicircle. Wherefore the two Angles *d a b* and *b a c* together will be equal to two Right ones, because they compleat the whole Semicircle, as two Right ones do (*Art.* 15.)

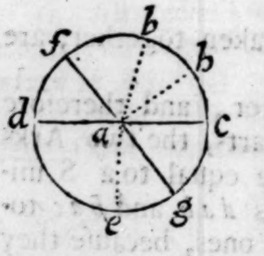
20. So that this Proposition is of Universal Truth, *That one Right Line falling on another makes the Contiguous Angles (together) equal to two Right ones.* For if the Lines are *Perpendicular* to each other as *p a* is to *d c*. then 'tis plain the Angles must be Right (by the 15.) And if the Line fall *Obliquely* as *b a* doth: then indeed the Angles are Unequal; But as much as the Obtuse one *d a b* exceeds one Right Angle, by so much is the Acute one *b a c* exceeded by the other Right one. So that the Smallness of one is compensated by the greatness of the other.



21. If two Angles which have one side Common to both, do make Angles equal to two Right ones, their other Sides do make but one Right Line. Let the Angle *d a b* and *b a c* be (*together*) equal to two Right ones. Then I say that the Lines *d a* and *a c* do joyn so together, as to make one Right Line (*vid. Fig. in Art.* 17.) which is clear from what hath been said. For if on the Center *a* you describe a Circle *d b c e*, the two Arks *d b* and *b c* will be equal to a Semi-Circle, because the two Angles *d a b* and *b a c* are supposed to be Equal to two Right ones. Wherefore the Lines *d a* and *c a* will make a

Diameter, and consequently be joined into one Right Line.

22. If from the Point *a* you draw several Lines, as *ad*, *af*, *ab*, *ah*, *ag*, &c. they will make diverse Angles; and all those Angles taken together, be they more or less, will be equal to four Right ones. For 'tis clear all these Angles together do compleat the Circle *dbce*, whose Circumference they divide into as many Arks as there are Angles. Now all these together are equal to four Quarters of a Circle; which is as much as to say that all the Angles are equal in the whole to four Right ones; for so many Right Angles do compleat the Circle.



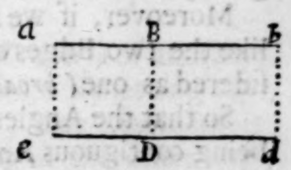
23. The Vertical or Opposite Angles are Equal. Let there be two Right Lines *dac* and *bae* (*crossing or cutting one another in the Point* a:) I say the Angle *dae* is Equal to *bac*. For the Ark *db* with the Ark *bc* makes a Semicircle: and so doth the same Ark *bd* with the Ark *de*. Therefore the Ark *bc* must be equal to *de*; because the Ark *db* continues the same whether it help to compleat the Semicircle with *de* or *bc*: (*Wherefore being taken away from both, it must leave the Ark* de = bc. *But if the Arks be equal the Angles subtended by them must be so too, and therefore the Angle* dae *is Equal to* bac.) and by the same Reason the Angle *dab* will be Equal to *eac*.

24. The Circumference of every Circle is (*supposed to be*) divided into 360 equal Parts, which are called Degrees: And every Degree into 60 Minutes, every Minute into 60 Seconds, every Second into 60 Thirds, and so on Infinitely. And to determine the Quantity of any Angle, we compute the Degree that (*the Ark which is its Measure*) doth contain. *v. gr.* When we speak of an

an Angle of 90 deg. we mean a Right Angle; because the Right Angle contains the fourth Part of the whole Circumference which is 90 deg. the fourth Part of 360. So an Angle of 60 deg. is an Angle that contains two Thirds of a Right one.

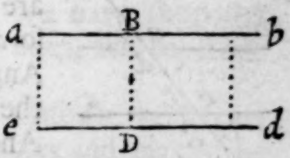
25. (*Degrees are marked either with Degr. or usually with a small Cypher over the last Figure as 60°.*) Minutes with a small Line as 50′, Seconds with two such, as 30″ Thirds with three such, as 25‴, &c. So that 25° 32′ 43″ is to be read 25 Degrees, 32 Minutes, 43 Seconds.

26. Two Lines are said to be *Parallel*, when they run always equi-distant from each other. Thus the two Lines *a b* and *e d* are *Parallel*, if they are equally distant from each other in *a e*, in *B D*, in *b d*, and in all other *Places*.

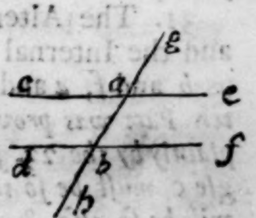


27. This Distance is always measur'd by a Perpendicular; as if from the Point *a* you imagine the line *a e* to fall perpendicularly on *e d*; as also doth the Line *b d* on the same Line; we naturally conceive that if those two Perpendiculars are of the same Length, or equal; the two Lines *a b* and *e d* are equally distant from each other in those two places, which is self-evident and needs no Proof.

28. Two Parrallel Lines being continued infinitely, yet can never meet. For being always equally distant, there may any where be drawn between them a Perpendicular equal to *a e* or *b d* and consequently they can never meet.



29. If a Line cut or cross two other Lines that are Parallel, it will be equally inclined to them both: And if a Line cutting two other Lines, be equally inclined to them both; Those two Lines are Parallel.



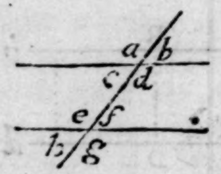
Let the two Parallel Lines be *c a e* and *d b f*, which are cut by the Line

g a b h. I say that that Line *g a b h*, is equally inclined to *c a e* as it is to *d b f*; that is, the Angle *g a e* is equal to the Angle *g b f*, which will be clear to any one that considers the thing; For if the Angle *g a e*, for Instance, were greater than *g b f*, and if the Line *a e* were opened or parted farther from *a g*; then must the Point *e* of the line *c e* incline or approach towards *f*; because *b f* is supposed not to part or open from *g b*, as *a e* doth from *a g*. Therefore the two Lines *c e* and *d f* will not be Parallel, (*which contradicts the Supposition:*) *'Tis the same thing if the Angle* g a e *should be said to be less than* g b f, *for then the Lines could not be Parallel.*

Moreover, if we imagine the two Parallel Lines to be like the two Edges of a Ruler, that Ruler may be considered as one (*broad and*) indivisible Line.

So that the Angles *h b d* and *c a g* may be considered as being contiguous Angles, and equal to two Right ones (by 20:) And the Angles *h b d* and *g a e* as Vertical or Opposite Angles (23.) and so equal to each other. But if *d b h* be equal to *g a e*, its equal *g b f* (23.) must be so too. *Wherefore the Parallel Lines have the same Inclination, or make equal Angles, with the crossing Line.*

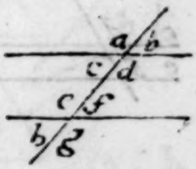
30. Whenever a Right Line cuts two Parallels it makes with them eight Angles: Of which four *a. b. h. g.* are *External*; and the other four *c. d. e. f.* are *Internal*. The Angles *c* and *f*, as also *d* and *e* are called *Alternate*. The Angles *e* and *a* as also *f* and *b* are called the *Internal* and *Opposite on the same side*. And the Angles *d* and *f*, as also *c* and *e* are called the *Internal Angles on the same side*.



31. The Alternate Angles, *c* and *f* as also *d* and *e*; and the Internal and Opposite ones on the same side, as *b* and *f*, *a* and *e* &c. are severally equal. (*The Latter Part was proved in Art.* 29. *and the former follows plainly by the* 23. *for since* b *is equal to* f, *its Opposite Angle* c *must be so too, and if* a *be equal to* e *its Opposite* d *will be so also*, &c. (vid. Art. 29. the latter Part.)

32. When

32. When a Line falls on two Parallel ones, it makes the Internal Angles on the ſame Side equal to Two Right ones.



I ſay the Angle *d* with *f*, is equal to two Right ones: Becauſe *f* is equal to *c* (by 31) and *c* and *d* together are equal to two Right ones (by 20.) Therefore *f* and *d* together muſt be equal to two Right ones, which was to be proved.

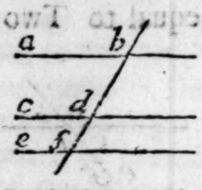
(*The ſame way may* c *and* e *together be proved Equal to two Right ones; for* c *and* d *taken together are ſo (by* 20) *but* d *is equal to* e (31) *Therefore* c *and* e *are equal to two Right ones.*)

33. One Propoſition is called the *Converſe* of another; when after a Concluſion is drawn from ſomething *Suppoſed*, in the *Converſe* Propoſition, that *Concluſion* is *Suppoſed*; and then that which was in the other *Suppoſed*, is now drawn as a Concluſion from it. For Example: We ſay here, if two Lines are Parallel, (*and another Croſs them,*) the Angles *d* and *f* together are Equal to two Right ones: Where we ſuppoſe the Lines to be Parallel and from thence conclude thoſe Angles muſt be Equal to two Right ones: But the Converſe is Thus; If the *Internal Angles on the ſame ſide d* and *f* together are Equal to two Right ones, then thoſe Lines are Parallel: Where after we have ſuppoſed theſe Angles equal to two Right ones, we conclude the Lines to be Parallel.

34. Converſe Propoſitions in this Caſe are very true, as, that if a Line cut two other Lines and makes the Alternate Angles equal; Thoſe two Lines are Parallel.

35. If two Lines are Parallel to a third Line they are ſo to one another.

Let the Line *a b* be Parallel to *c d*; and let *e f* alſo be Parallel to the ſame Line *c d*, I ſay *a b* is Parallel to *e f*: For if you draw a Line as *b d f* cutting them all Three; the Angle *b* will be equal to *d* (by 31.) and the ſame *d* will be equal to *f* (by 31.) becauſe *e f* is alſo Paral-



Parallel to *c d.* Wherefore the Angle *b* must be equal to *f*: Because 'tis an Axiom, *That if two things are equal to a third they are so to one another.* But if the Angle *b* be equal to *f*, then the Line *a b* is Parallel to *e f* (by 34.)

BOOK II.

Of Triangles.

1. A *Figure* is a Space compassed round on all Sides. And if the Lines which Terminate it are all Right ones 'tis called a *Rectilineal* (or Right Lined) *Figure*: If they are Crooked 'tis called a *Curvilineal*; and if they are partly Right Lines and partly Crooked, 'tis called a *Mixt* Figure.

2. There are *Plane* Figures, which are *Plane Surfaces*, and there are *Solid* ones, which have three Dimensions. But we speak here only of Plane Surfaces, or Plane Figures.

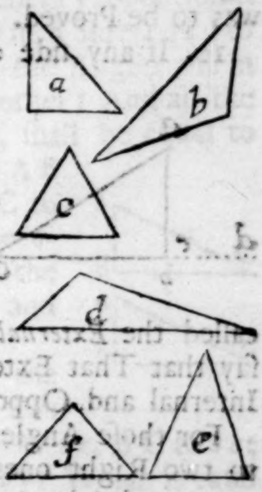
3. All the Lines which encompass any Figure taken together, make that which is called the Circumference, Perimeter, or the Compass of the Figure.

4. Of all Curvilineal or Mixt Plane Figures, in Common Geometry we consider properly only the Circle, or a Part of a Circle terminated on one side by an Ark, and on the other by one or more Right Lines.

5. Of Rectilineal Figures the most Simple are *Triangles* which are Terminated by three Right Lines (*and no more*) making as many Angles.

6. A

6. A Triangle as *a* which hath one Right Angle is a *Right-angled* Triangle; if it have one Angle Obtuse 'tis called an *Obtuse-angled* One, as *b*; and if all its three Angles are Acute, 'tis called an *Acute-Angled* Triangle, as *c*.

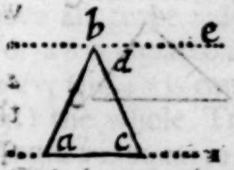


7. If a Triangle have all its three sides Unequal 'tis called a *Scalene* as *d*. If it hath two sides equal 'tis called an *Isosceles* as *e*, And if all the three sides are equal, 'tis called an ***Equilateral*** one, as *f*.

8. When two sides of a Triangle are considered, they may be called its *Legs*, and the third side may then be called the *Base*. But any one side may be called the *Base*, (*tho' we usually and most properly call that so, which lies Parallel to the Horizon, and which is next to us.*)

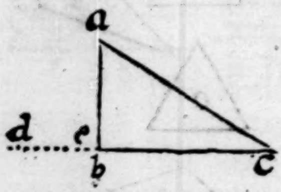
9. In every Triangle, the three Angles taken together, are Equal to two Right ones.

Let the Triangle be *a b c*: I say, that the Angle *a* added to the Angle *c*, added to the Angle *b* (*or the Sum of all three*) are equal to two Right ones. For let *d e* be drawn Parallel to the *Base a c*, then will those two Parallel-Lines be cut by the Line *b c*; and consequently the Alternate Angles *c* and *d* will be equal to each other (by 1. 31.) Moreover the Line *b a* falling on or cutting the same Parallels *d e* and *a c*, will make the Internal and opposite Angles on the same side equal to two Right ones; that is, *a* added to *a b e* are equal to two Right Angles (by 1. 32.) But the Angle *a b e* contains the two Angles *b* and *d*. So that the Angle *a* added to *b* added to *d* will be equal to two Right ones. But *c* being equal to *d*, it will follow that (*a* added to *b* added to *c*; or the Sum of all three together



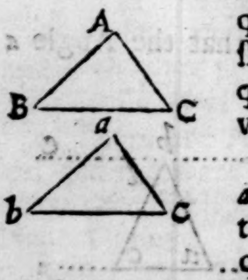
together must be equal to two Right ones: which was to be Proved.

10. If any side of a Triangle be Produced, or drawn out, the External Angle will be equal to the two Internal and Opposite Angles, (*taken together*) Let the Triangle be *a b c*, whose Base *c b* draw out to *d*, by which means a new Angle as *e* will be made, which is called the *External* Angle of that Triangle. Then I say that That External Angle *e*, is equal to both the Internal and Opposite ones *a* and *c*.



For those Angles *a* and *c* together with *b* are equal to two Right ones (by the Precedent,) and so also are *e* and *b* by (1. 20) wherefore *e* must be equal to *a* added to *c*, because together with *b*, it makes two Right Angles, as they do. Q. E. D.

11. If a Triangle A B C hath two Sides A B and A C equal to two others *a b* and *a c* in another Triangle, and if also the Angle A be equal to *a*, I say the Third Side B C shall be equal to *b c*; the Angle B equal to *b*, the Angle C to *c*, and the whole Triangle A B C to *a b c*.

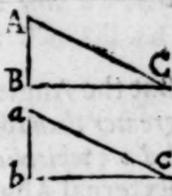


For if we imagine the Triangle *a b c* to be placed upon A B C, so that the side *a b* shall lie exactly on its equal A B: Then must the Side *a c* fall on its equal A C because the Angle *a* is equal to A, and so the Point *c* will fall on C, and *b* upon B, and the whole Triangle *a b c* on the A B C: because all things do so exactly answer, that nothing of the Upper Triangle, can fall besides the Under one.

12. Figures which do thus meet, fit, or answer to each other exactly, when they are placed one upon the other are called *Congruous* Figures, *Quia mutuò sibi congruunt.* And it is an universal Axiom, *Quæ sibi mutuò congruunt sunt æqualia*; *i. e.* Those Figures which placed one upon another do Answer to one another exactly, are equal.

13. The

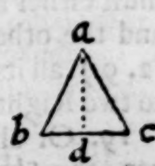
13. The *Converse* also of the Precedent Proposition is true: That is, if a Triangle hath all its 3 Sides equal to the 3 Sides of another Triangle, all the Angles also in one shall be equal to those in the other: And all the Space which one Triangle contains, shall be equal to that contained in the other: As if A B be equal to *a b*, A C to *a c*, and B C to *b c:* I say that the Angle A shall be equal to *a*, B to *b*, and C to *c*; and the whole Triangle A B C, to *a b c:* and this needs no other Proof.



14. If the Angle A be equal to *a*, the Angle B to *b*, and the Side A B to *a b:* Then shall the Side A C be also equal to *a c*, B C to *b c*; and the whole Triangle A B C to *a b c:* Which is easie to Prove by the precedent Propositions.

15. In every *Isosceles* Triangle, the Angles at the Base, opposite to the equal Legs, are equal.

Let the Triangle be *a b c*, whose Legs *a b* and *a c* are equal: I say the Angle *b* is also equal to *c*. For Imagine the Base *b c* divided into two equal Parts in the Point *d*, then will the Line *a d* (*which let be drawn*) make of the whole, two Triangles, *a b d* and *d a c*, which will have all three Sides in one equal to those in the other: For *a b* is equal to *a c* by the Supposition, and *b d* is equal to *d c*, and *a d* is common to both. Wherefore (by 2. 13.) the whole Triangle *b a d* is equal to *d a c* and the Angle *b* is equal to *c*; which was to be proved.

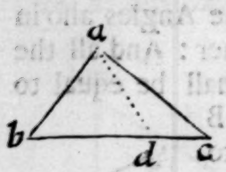


16. In an *Isosceles* Triangle if a Line drawn from the Angle at the top do (*Bissect or*) divide the Base into two equal Parts, it is both Perpendicular to the Base, and also bissects the Angles at the Top into two equal Parts. For (*vid. Fig. præcedent*) the Angle *a d c* is equal to the Angle *a d b* (by the last) and consequently they must be both *Right ones*; and therefore the Line *a d* is Perpendicular to the Base *b c* (1. 15.) and the Angle *d a c* will be equal to *d a b* (by the last Prop.)

17. In every Triangle the Greater side is always opposite to, or *subtends* the Greater Angle.

I

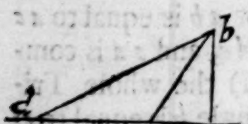
In the Triangle abc let the Side bc be longer than ba; then I say the Angle bac subtended by the greater Side bc is bigger than the Angle c, which is subtended by the lesser Side. For let bd be taken equal to ba, then will abd be an *Isosceles* Triangle; whose Angle bad will be equal to bda (2. 15.)



But the Angle cab is bigger than bad; (*the whole being greater than the Part*) and therefore must be bigger than bda (*which is equal to* b a d). Now the Angle adb is an External Angle in Respect of the little Triangle adc; and therefore must be bigger than the Internal one c (by 2. 10.) Wherefore the Angle bac being bigger than d, must certainly be bigger than c; which was to be Proved.

18. Every Triangle must necessarily have two Acute Angles: For if it had but *one*, the other two Angles must either be both Obtuse, both Right, or one Obtuse and the other Right: None of which can be, because (2. 9.) all its three Angles taken together are equal but to two Right ones.

19. Of all Lines that can be drawn from a Point given, the *Shortest* is the Perpendicular; and they are all *Longer* according as they are farther distant from it. Let the given Line be ad, and the Point given b; let ba be Perpendicular to da; let also bc and bd be drawn. I say that ba is the shortest Line that can possibly be drawn from b; and (for instance) is shorter than bc (*or any other that can be assigned*) and I say also that bd is longer than bc.



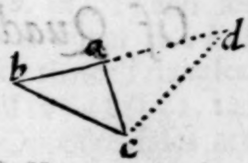
For in the Triangle bac, the Angle a is a *Right one*, and consequently bigger than either of the other; because they must necessarily be both Acute (by the last Prop.) Therefore the Side bc is longer than ba (2. 17.) as subtending a greater Angle.

So

So alſo in the Triangle *d b c*, the Angle *d c b* is *Obtuſe*, becauſe the Angle *b c a* is *Acute*: And conſequently the Side *d b* muſt be longer than *c b a*, as ſubtending a greater Angle (2. 17.)

20. In every Triangle any two Sides taken together are longer than the Third.

Let the Triangle be *a b c*; I ſay that the ſide *a b* added to *a c* are longer than *c b*: For produce *b a*, till *a d* be equal to *a c*; then will the Triangle *a d c* be an *Iſoſceles*, and conſequently the Angle *b d c* will be equal to *a c d*. (2. 15.)



Therefore the Angle *b c d* (which is greater than its part *a c d*) will be bigger than *d*. And if you now conſider the Figure as but one Triangle *b d c*, then will the Side *b d* be longer than *b c*, becauſe (2. 17) ſubtending a greater Angle. But *b d* is equal to the two ſides of the former Triangle *b a*, and *a c*, becauſe *a d* was taken equal to *a c*. Wherefore the two ſides *b a* and *a c* together are longer than the *Third b c*. which was to be proved.

21. Although this Propoſition be thus demonſtrated, yet it might paſs for a thing diſcoverable at firſt Sight. For the Line *c b* being a *Right* Line, runs the neareſt of all poſſible ways between the Point *c e* and the Point *b*. So that all others (*that do not go the ſame way and fall in with it*) muſt go about, and conſequently a longer way, as *c a b*, *c f b*, *c d b*, &c.

And we may with *Archimedes* lay it down as a Principle, That thoſe Lines which fetch Compaſſes and go about are always longeſt, becauſe they include others that are ſhorter. So *c d b* is longer than *c e b* and *c a b* than *c d b*. Provided that the Lines don't interfere with each other and make many Angles, as thoſe in *f* of this Figure; for then 'tis poſſible they may in the whole be longer than *c a b*, tho' included within the compaſs of *a b d*.

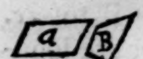
BOOK

BOOK III.

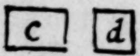
Of Quadrilateral Figures and Polygons.

1. THose Figures whose Sides are four Right Lines and those making four Angles, are called *Quadrilateral* or four sided Figures.

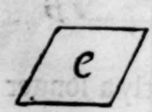
2. When the opposite Sides are Parallel, the Quadrilateral Figure is called a *Parallelogram* as *a*; but if not 'tis called a *Trapezium* as B.



3. When the Parallelogram hath all its four Angles Right, 'tis called a *rectangled Parallelogram*; or for brevity sake a *Rectangle* as *c*: And if the Angles are right and the Sides are all equal 'tis called a *Square* as *d*.



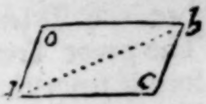
4. If a *Parallelogram* hath all its Sides equal, but its Angles unequal, then 'tis called a *Rhombus* as *e*.



5. If a *Parallelogram* hath neither its Angles nor sides all equal tis, 'called a Rhomboides as *a*.

6. In every *Parallelogram* the opposite Angles are equal. Let the Parallelogram be *o b c d*. I say the Angle *o* is equal to *c*, for the Angle *o* is equal to the Alternate one *b* (1. 31.) and the External one *b* is equal to the Internal one *c* (1. 31.) wherefore *o* is equal to *c*.

7. A



7. A Line as *d b* drawn across the Figure from Angle to Angle is called the *Diagonal*, and by some, the *Diameter*.

8. Every Parallelogram is divided into two *Equal Parts* by the Diagonal. The Diagonal *b d* divides the Parallelogram *o b c d* into the Two Equal Triangles *o b d* and *b c d*. For, 1. The Angle *o* is equal to *c* (3. 6) 2. The Angle *o b d* is equal to *c d b*. (1. 31) and for the same Reason, the Angle *o d b* is equal to *c b d*; and the side *b d* is common to both these Triangles, wherefore the Triangle *o b d* is equal to *c d b* (by 2. 14)

9. In every Parallelogram The Opposite sides are always equal.

For, (*drawing the Diagonal* d b) the whole Triangle *d o b* will be equal to the Triangle *b c d* by the foregoing Prop. And Consequently the side *c d* must be equal to *o b* and the side *o d*, to *c b*.

10. Two Diagonals *a c* and *b d* do bissect each other in the Middle at *e*.

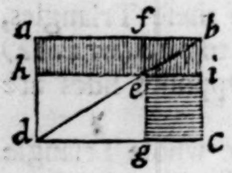
For in the Two Triangles *a e d* and *b e c*, The side *a d* is equal to *b c* (3. 9) The Angle *e a d* is equal to *e c b* (1. 31) and also the Angle *a d e* is equal to *c b e* (1. 31.) and moreover the (*Vertical*) Angles *a e d* and *c e b* are equal also (1. 23) *Wherefore* (*the whole Triangle* a e d *is equal to the Triangle* b e c (2. 14.) and consequently the side *d e* is equal to *e b*, and the side *a e* to the side *e c*. The two Diagonals therefore bissect each other Q. E. D.

11. Every Right Line as *f g* passing by the middle of a Diagonal divides the Parallelogram into two Equal Parts.

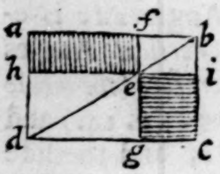
To demonstrate which, the *Trapezium* or Irregular Quadilateral Figure *f g d a* must be proved equal to the *Trapezium* *f g c b*. And that is thus done. 1. The Triangle *b e f* is equal to the Triangle *d e g*: for the side *d e* is equal to *e b* by the Supposition; and the Angle *e f b* is equal to *e g d*

(1. 31) and the opposite Angles at *e* are equal, wherefore the Triangle *efb* is equal to *edg*. (2. 14). 2. The great Triangle *abd* is equal to *bdc* (3. 8) wherefore if from the Triangle *abd* you take away the little Triangle *feb*, and instead of it put the Triangle *edg* (*which is equal to* feb) you will have the *Trapezium* *fadg*, which will be equal to the Triangle *adb*: That is to just one half of the whole Parallelogram (3. 8) Which was to be proved.

12. If in the Diagonal *db* you take a Point as *e*, and thro' it draw two Lines *hi* and *fg* Parallel to the two sides of the Parallelogram, It will be divided by them into four lesser *Parallelograms*, *i.e. febi*, *hegd*: (which two are called the *Parallelograms about the Diameter*) and *afhe*, *eicg*: which other two are called the *Complements*. And those two Complements with either of the Parallelograms about the Diameter make a Figure that is called a *Gnomon*. As you see in the Figure; where the Gnomon is distinguished by being Shaded.



13. In every Parallelogram the Complements are Equal. We must Prove that *ehaf* is equal to *egci*.

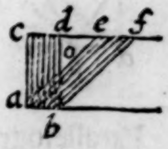


Demonstration: The whole Triangle *abd* is equal to the whole *bdc*. (3. 8)

And the Little Triangle *efb* is (for the same Reason) equal to *ebi*. And the Triangle *hed* is also (by the same) equal to *edg*. Wherefore if from the two Equal Triangles *abd* and *bdc* we take away equal things, *viz*. if from one we take away *efb* and *dhe*, and from the other *ebi* and *egd*, There will remain on one side the Parallelogram *ehaf*, equal to the Parallelogram *eicg* which remains on the other, which was to be proved.

14. Par

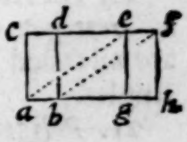
14. Parallelograms having the ſame Baſe and being between the ſame Parallels are Equal.



Let there be a Parallelogram *a b c d* and another *a e f b*, both on the ſame Baſe *a b*; and let the Line *c d* when produced be ſuppoſed to paſs by *e f*; So that the two Parallelograms ſhall be between the ſame Parallels and Terminated by them; that is, between the two Parallels *c f* and *a b*. I ſay then that the Parallelogram *a c d b* is equal to *a e f b*.

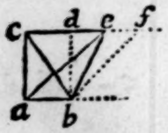
For 1. *c d* is equal to *e f*, becauſe both are equal to *a b* (3. 9.) Therefore if to each of thoſe equal Lines you add the Line *d e*, *c e* will be equal to *d f*: 2. *c a* Is equal to *d b* (3. 9.) 3. The Angle *a c e* is equal to *b d f* (1. 31.) wherefore the Triangle *c a e* is equal to *d b f*. (2. 11.) Now if from each of theſe equal Triangles you take away the white Triangle *d o e* that is between the Parallelograms, and add to them both the Triangle *a o b* there will ariſe the Parallelogram *c d b a* on one ſide, equal to the Parallelogram *a e f b* on the other. (*And they muſt be equal, becauſe when the white Triangle was taken away, there remained the Trapezium* c a o d *equal to* o b f e, *and when at laſt to each of them was added the Triangle* a o b, *the Sums muſt needs be equal. That is* c d a b *equal to* a e f b. Q. E. D.)

15. Parallelograms on equal Baſes *a b* and *g h* and between the ſame Parallels *a h* and *c f*, are equal.



For if we Imagine the third Parallelogram *e f b a* to be drawn; that ſhall be equal to the Parallelogram *a b c d*, becauſe on the *ſame Baſe a b* with it, and between the ſame Parallel Lines *a h* and *c f*. And that Parallelogram will alſo be equal to *e f g h*; becauſe it hath the ſame Baſe *e f*, with it (it matters not whether you reckon the Baſe above or below) and is between the ſame Parallels. Therefore *h f e g* and *b d c a* being both equal to the third Parallelogram *e f b a*, muſt be equal to each other.

16. *Triangles* on the ſame Baſe *a b* and being between the ſame Parallels *c f* and *a b*, are always equal.



The Triangle *a b c* is equal to *a e b*: Becauſe if you imagine a Line *b d* drawn Parallel to *a c*, and another as *b f* drawn Parallel to *a e*; there will be made two Parallelograms *a c d b* and *a e f b*; which being on the ſame Baſe and between the ſame Parallels, will be equal to one another (3. 14.)

But the Triangle *a b c* is the half of the Parallelogram *a c d b* and the Triangle *a b e* is the half of the Parallelogram *a e f b* (3. 8.) wherefore, (*ſince the wholes are equal, the Halves muſt*) and conſequently the Triangle *a c b* is equal to the Triangle *a e b*.

17. Triangles on *equal Baſes*, and between the ſame Parallels are alſo equal; as is every eaſie to prove from (3. 15.)

18. If a Triangle have the ſame Baſe with a Parallelogram, and be alſo between the ſame Parallels, it ſhall be juſt the half of that Parallelogram. For it will ſtill be equal to *a b c* which is juſt half (3. 8.) of *a c d b*.

19. A *Pentagon* is a Figure having five Sides and five Angles.

If all the Sides are equal, and conſequently the Angles, 'tis call'd a *Regular Pentagon*.

20. An *Hexagon* is a Figure of ſix Sides and Angles; an *Heptagon* of ſeven, an *Octagon* of eight, &c. which are all called *Regular* when they have equal Sides and Angles.

21. *Polygon* in the general ſignifies any Figure of many Sides and Angles; but no Figure is called by this Name, unleſs it have more than 4 or 5 Sides.

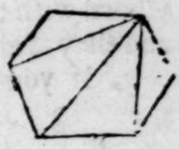
22. Every

22. Every *Polygon* may be divided into as many Triangles as it hath Sides, If any where within the *Polygon* you take a Point as *a* and from thence draw Lines to every Angle *a b*, *a c*, *a d*, &c. for they ſhall make as many Triangles as the Figure hath Sides.

23. The Angles of any *Polygon* taken all together, will make twice as many Right ones, except four, as the Figure hath Sides. *v. gr.* If the *Polygon* have 6 Sides; the double of that is 12, from whence take four, there remains eight. I ſay that all the Angles of that *Polygon*, *viz.* *b*, *c*, *d*, *e*, *f*, *g*, taken together, are equal to eight Right Angles. For the Lines *a b*, *a c*, *a d*, &c. do divide the Figure into ſix *Triangles*; the three Angles of each of which are equal to two right ones (2. 9.) ſo that all their Angles together make 12 right Ones. But now each of theſe ſix *Triangles* hath one Angle in the Point *a*, and by it they compleat the ſpace all round the ſaid Point. And all the Angles about that Point, are equal to four right Ones (1. 22.) Wherefore thoſe 4 being taken from 12 (*The Sum of the Right Angles of all the ſix Triangles*) leaves eight, the Sum of the right Angles of the *Hexagon*.

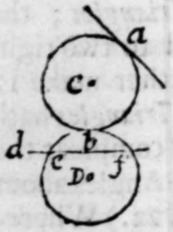
So that the Figure hath plainly twice as many right Angles as it hath Sides, except four; which was to be proved.

24. A *Polygon* may be divided alſo into *Triangles* by drawing Lines from Angle to Angle. But then the Number of the Sides will exceed that of the *Triangles*.



BOOK IV.

Of A Circle.



A Line is ſaid to *Touch* (*or to be a Tagent to*) a Circle, when tho' produc'd both ways from from the Point of Contact, it will only touch it, and not *cut* or enter within it. Thus the Line *a* Touches the Circle *C* as that Circle *C* doth the Circle *D*; but *d* enters within the Circle, and *cuts* it, *and is call'd a Secant*.

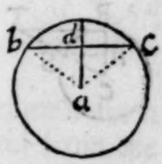
2. If a right Line enter within a Circle and cut it into two Parts, thoſe Parts are call'd *Segments*: *b* is a leſs Segment and D a greater: That (*part of the* Line) cutting the Circle (*which is within it*) is called a *Chord* as *e f*. And the Parts of the Circle (*or rather Circumference*) cut off, are called *Arks*: The Chord with the Ark makes two mixt Angles as *e* and *f*, and they are call'd *Angles of a Segment*.

3. If you take a Point as *c* in the Ark of any Segment and from thence draw two Lines *c a*, and *c b* (*to the ends of the Chord*) they ſhall make an Angle *a c b*; which is call'd *an Angle in a Segment*: And that Angle *a c b* is ſaid to *Inſiſt* or *ſtand* on *a b d* the Ark of the other Segment below.

4. A Sector of a Circle is a mixt Triangle comprehended between two Radius, *a b*, *a c*, and the Ark of the Circle *b c*, 'tis mark'd in the Figure by being shaded.

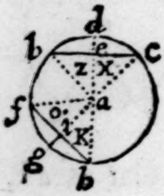
5. If to the end of any Radius, or Semidiameter, *a b*, you draw a Perpendicular as *d b*, it shall touch the Circle but in one Point. And all the Points of the Line *b d* shall be without the Circle. *v. g.* I say the Point *d* (or any other assignable) is without: For if you draw the Line *a d* from the Center, that shall cut the Circle in the Point *c*, that Line *a d* will be longer than *a b*; (2. 19.) and consequently longer than *a c* which is equal to *a b* (1. 14.) Wherefore the Point *d* is without the Circle. Q. E. D.

6. *A Chord*, as *b c* is divided into two equal Parts (or Bissected) by a Perpendicular *d a*, drawn from the Center *a*. For the Triangle *a b c* is an *Isosceles*, because *b a* is equal to *c a* (1. 14.) and therefore the Perpendicular *a c* Bissects the the Base *b c* (2. 16.) The Ark *b c* is also by this means Bissected.

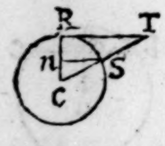


7. If two Lines *c b* and *c d* (*drawn from the same Point without*) Touch a Circle, they are equal one to another. For draw from the Center to the Points of Contact, *b a* and *a d*. Then will those Lines be Perpendiculars to the Tangents (by 4. 5.) Then if you draw also the Line *b d* the Angle *a b d* will be equal to *a d b*. (2. 15) Wherefore if from the Right and (*consequently*) equal Angles *c b a* and *c d a*, you take away the Equal ones *a b d* and *a d b*, the remaining Angles *c b d* and *c d b* will be equal: Wherefore their opposite sides must also be equal (*by the converse of* (2. 15.) That is, *c b* is equal to *c d*. Q. E. D.

8. Equal Chords as *b c* and *f h*, do cut off equal Segments *b d c* and *f g h*. And the Perpendiculars *a e* and *a i*, drawn to them from the Center are also equal, as is easily proved; (saith *Pardie*, *But he give us no Demonstration.*) *Yet 'tis plainly thus prov'd: The Chords and Arks are both Bissected by the Perpendiculars* (4. 6.) *And therefore the Sectors* c a d, d a b, f a g, and g a h *must be all equal; and so also will all the Triangles* x, z, o *and* k *be.* (2. 11.) *Therefore their Doubles will also be equal* i.e. *the Sector* b a c *will be equal to* f a h: *And the Triangle* b a c *to the Triangle* f a h. *And if these last Triangles are taken from the equal Sectors* h a f *and* b a c *the Segments* b d c *and* h g f *must remain equal. That the Perpendiculars are equal, is plain from the equalities of the Triangles* z *and* o, *or* X *and* K.



9. Let there be a Semidiameter *Rc*, a Perpendicular (*to it without the Circle*) *RT*, another Line cutting the Circle in *S*, and a Perpendicular (*let fall from thence*) to the Radius *RC* in *n* (*a Point within the Circle*). All these Lines have Artificial Names. The Line *TR* is called the *Tangent* of the Ark *RS* (*which suppose*) 30°. *TC* is call'd the *Secant* of the same Ark of 30° and the Line *Sn* is called the (*Right*) *Sine* of the same Ark. *R C* is by some called the *whole Sine*, but most usually the *Radius*.



10. If in the Circumference of a Circle, you take two Points as *a* and *b* and from thence draw two Lines to the Center *c*, and two others to any Point, as *d* in the Circumference; they will make two Angles, of which *a c b* is called an *Angle at the Center* and *a d b* an *Angle at the Circumference.*

11. The

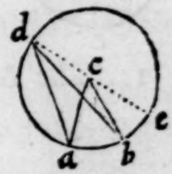
11. The Angle at the Center *a c b* is always double to one at the Circumference *a d b* (*insisting with it on the same Ark* a b.)

Of which there are three Cases.

1. If one of the Lines as *d b* pass thro' the Center *c*, then 'tis plain the external Angle *a c b* (2. 10.) will be equal to both the Internal and Opposite ones *a* and *d* taken together.

But the two Angles *d* and *a* are equal, because *a c d* is an *Isosceles* Triangle, whose side *a c* is equal to *c d* (2. 15.) Therefore the Angle *c* at the Center being equal to both, is double of either alone: That is double to *d*. Q. E. D.

2. If neither of the Lines *d b*, *d a*, (*which form the Angle at the Circumference*) pass thro' the Center *c*: (*But fall both on the same side of the Diameter*) Let the Diameter *d c e* be drawn. Then will the whole Angle *a c e* (*at the Center*) be double to the Angle *a d e* (at the Circumference) by what was proved in the first Case. Also the Angle *b c e* is double to *b d e*, by the same. Wherefore if from the Angle *a c e*, we take away that *b c e*, and from the Angle *a d e* which is the half of *a c e*, we take away *b d e*, which also is the half of *b c e*, the remaining Angle *a d b* must be just the half of *a c b*. For 'tis as plain as an Axiom, that if one Quantity be double to another, and you take away from the bigger just the double of what you take from the other, the remainder of the Bigger must be double to the remainder of the Lesser.



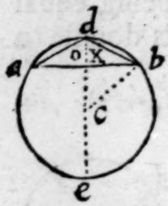
3. If the Diameter fall between the Lines forming the Angle at the Circumference. Then will, as before, the Angle *a c e* be double to *a b e* (*by Cas.* 1 *of this*) and the Angle *e c d* will be double to *e b d* by the same; therefore the whole Angle *a c d* must be double to *a b d*. (*So that in all Cases, the Angle at the Center is double to one at the*

be Circumference if they both stand on the same Ark, or (which is all one) are in the same Segment.

12. All Angles (*in the same Segment or*) Insisting on the same Ark *a b* are equal, let them Terminate in any part of the Circumference whatsoever.

For the Angle *a d b* will be equal to *a e b* because each is the half of the Angle at the Center *a c b* (4. 11.)

13. An Angle at the Center *b c e* standing on half of the Ark *a e b*, is equal to the Angle *a d b* at the Circumference standing on the whole Ark. (*For* c *is equal to twice* x; (by 4. 11.) *and* x *is equal to* o *that is to half* a d b (4. 6 *and* 4. 8.) *Wherefore* c *is equal to* a d b. Q. E. D.

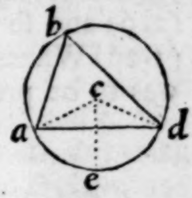


14. The Angle *a d b* standing on the Semicircumference *a b e* (*or being in the Semicircle* a d b) is a right one. Let *c e* be drawn Bissecting the Semicircumference; then is (by the *Precedent*) the Angle *a c e* at the Center, standing on half a Semicircle (*or on a Quadrant*) equal to *a d b* at the Circumference, which stands on twice the Ark. But *a c e* is a right Angle, wherefore *a d b*, (its equal) must be so.

15. The Angle *a b d* in a Segment less (*than a Semicircle*) is *Obtuse*: Because the Ark *a e d* being more than half the Circumference, its half, the Ark *a e*, must be more than 90*d*, therefore the Angle *a b d*, which is equal to *a c e*, (4. 13.) must also be more than 90*d* that is Obtuse.

16. The

16. The Angle *a b d* made in a Segment greater than a Semicircle, is *Acute.*



For 'tis equal to the Angle *a c e.* (4. 13.) whose Measure *a e* being the half of *a e d*, an Ark less than a Semicircle, must be less than 90*d*. And therefore *a b d* is less than 90*d* (1. è) Acute.

17. If a right Line as *g b* touch a Circle, as in the Point *a*; and another Line as *a e* cut it there. The Angle *b a e* shall be equal to *h*, or any Angle made in the opposit Segment *a h e.* And the Angle *e a g* shall be equal to *f*, or any Angle made in the other Segment, *e f a.*

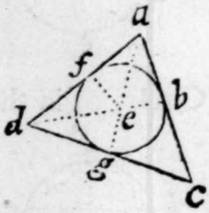
For, drawing the Diameter *a d* which will be Perpendicular to *a b* (4. 9.) *(and also the Line* d e:) the Angle *a e d* will be a right one; (4. 14.) and consequently, because the three Angles of every Triangle are equal to 2 right ones, (2. 9.) the Angle *e a d*, tegether with *d* must make just another right Angle.

But that Angle *d a e* together with *e a b* doth make also a right one, because the Radius *c a* is Perpendicular to the Tangent *a b*; wherefore *(take away* e a d *from both)* and then *e a b* will remain equal to *d*. And consequently to *h*, or to any other Angle *(in that Segment* a h e*)* or that stands on the same Ark *e f a*. For all those Angles are equal (by 4. 12.) The Angle *e a b*, therefore, is equal to *h*: Which is the first part of the Proposition.

We must next prove the Angle *g a e* to be equal to *f*; which is the other part.

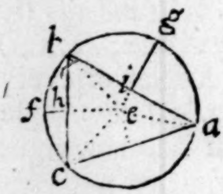
In the Triangle *a f e* all the three Angles *e*, *f* and *a* are equal to 2 right ones (2. 9.) And the Angle *e* is equal to *f a b*; by the first part of this Proposition, for *f a* may be consider'd as cutting the Circle in the Point *a* where *a b* touches it, and consequently *f a b* will be equal to any Angle that can be made in the opposit Segment *a b d e f*; and

and therefore to *e*. Now the two Angles *e a f* and *f a b* (that is *e*) together with *f*, are equal to two Right ones (2.9) and ſo are *e a f* and *f a b* taken together with *g a e* (1.20) Wherefore the Angle *f* is equal to *g a e*. Which was to be proved.



18. A Rectilineal Figure is ſaid to be *Circumſcribed* about a Circle, when all its Sides touch the Circle without cutting it. Thus the Triangle *d a c* is Circumſcribed about the Circle *b g f*; becauſe every ſide of the Triangle Touches the Circle, in *b*, *g*, and *f*.

19. A Figure is ſaid to be *Inſcribed* in a Circle when all its Angles are in the Circumference of that Circle, as the Triangle *a b c* in the following Figure.



20. Every Triangle, *a b c* may be Inſcribed in a Circle; for if two Lines as *e h* and *e i* are drawn Perpendicularly Biſſecting the ſides *b a* and *c b*, they will croſs or meet each other in the Point *e* on which as on a Centre, a Circle may be drawn, which ſhall paſs thro' *b*. And I ſay alſo that That Circle ſhall paſs thro' *a* and *c*.

For 1. The two Triangles *e i b* and *e i a* are equal; becauſe *i b* is equal to *i a* by the ſuppoſition, the ſide *e i* is common to both, and the Angles at *i* are Right. Wherefore the ſide *e b* is alſo equal to *e a* (2. 11)

2. And for the ſame Reaſon (*The Triangles* e h c *and* h b e *may be proved equal*) and conſequently the ſide *e c* alſo will be equal to *e b* and to *e a*. But if thoſe three Lines are all equal, the Point *e*, where they meet, muſt be the Center of a Circle of which they are *Radii*: And therefore the Triangle is Circumſcribed by a Circle. Q. E. D.

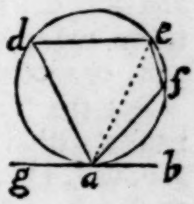
21. Every Triangle may (*have a Circle inſcribed in it, or*) be Circumſcribed about one. *vid. Fig.* 18.

For

For drawing the Lines *ae* and *ed* Biſſecting the Angles *a* and *d*, and from the point *e* where they croſs, letting fall the Perpendiculars (to the ſides of the Triangle) *eb*, *ef* and *eg*: I ſay that if you draw a Circle on the Center *e*, thro' *b*; that Circle ſhall touch all the ſides of the Triangle in the points *b*, *f*, and *g*. For 1. The two Triangles *aef* and *aeb* are equal, as having the ſide *ae* common, the Angles at *f* and *b* right, and thoſe at *a* equal (by the Suppoſition:) wherefore *eb* is equal to *ef*. (2. 14)

2. By the ſame Method, *eg* may be proved equal alſo to *ef*, (that is to *eb*) ſo that theſe three Lines being all equal, a Circle will paſs thro' their three extremities, of which Circle they will be Radii; and being alſo all perpendicular to the ſides of the Triangle, the ſaid ſides are Tangents to that Circle (4. 5) and therefore do *Circumſcribe* it (by 4. 18.)

22. Every Quadilateral Figure *defa* Inſcribed in a Circle hath its two Oppoſite Angles taken together (as *d* added to *f*) equal to two Right Ones.



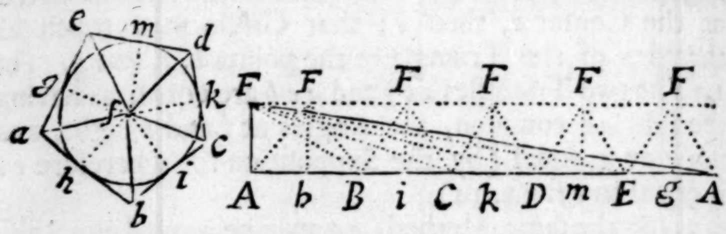
For if thro' the Point *a*, there be drawn a Tangent as *gb*, and a Diagonal as *ea*: The Angle at *f* will be equal to *gae* (4. 17.) and the Angle *eab* will be equal to *d* (4. 17) And conſequently the two Angles *gae* and *eab* being equal to two Right ones (1. 20) the Angles *d* and *f* taken together muſt be ſo too.

After the ſame manner might the other two oppoſite Angles *daf* and *def* be proved equal to two Right ones; By drawing another Tangent through the Point *f*.

23. The Converſe of this Propoſition is alſo manifeſt; *viz*. That if any Quadilateral Figure have its Oppoſite Angles equal to two Right ones, it may then be inſcribed in a Circle, That is, a Circle may be made that ſhall touch or paſs thro' all its four Angular Points.

24. Every Polygon circumſcribed about a Circle, is equal

equal to a Rectaangled Triangle, one of whose Legs shall be the Radius of the Circle, and the other the *Perimeter* (or the Sum of all the sides) of the Polygon.



Let the Line F A be equal to the *Radius f h*, and to it at Right Angles draw the Infinite Line A B C D, &c. out of which take A *h* equal to *a h*, *h*B equal to *h b*, B *i* equal to *b i* and *i* C equal to *i c* &c. So that the whole Line A B C D E A may be equal to the whole Compass or *Perimeter* of the Polygon *a b c d e a.* Also draw F F parallel to A A, So that all the Perpendiculars F *h* F *i* F *k*, &c. may be equal to the Radius *f h* or *f i*, &c. 'Tis then Plain that the Triangle A F B will be equal to the Triangle *a f b* in the Polygon, and the Triangle B F C to *b f c* and also C F D to *c f d*, &c. so that all these Triangles taken together will be equal to all these in the Polygon, or to the whole Polygon.

But the Triangle F A A is equal to all the five Triangles within the Parallels; because drawing the Lines, B F, C F, D F, &c. The Triangle F A B will be equal to F A B, F B C to F B C, &c. (3. 16) wherefore the Triangle F A A is equal to the Polygon, which was to be proved.

25. Every Regular Polygon is equal to a Rectangle Triangle, one of whose Legs is the Perimeter of the Polygon, and the other a Perpendicular drawn from the Center, to one of the sides of the Polygon. The Proof of which is the same as that in the precedent Proposition; For all the Perpendiculars *f h*, *f i*, *f k*, &c. are equal &c. See the last Figure.

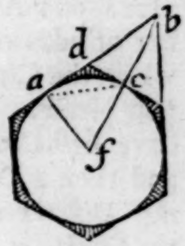
26. Every Polygon Circumscribed about a Circle is bigger than it; and every Polygon Inscribed is less than the

the Circle. As is manifeſt, becauſe the thing containing is always greater than the thing contained.

27. The *Perimeter* or (as ſome call it tho' Improperly) the Circumference of every Polygon Circumſcribed about a Circle, is greater than the Circumference of that Circle; and the Perimeter of every Polygon Inſcribed is leſs, as is plain from the 21ſt of the 2d Book.

28. If in any little Segment of a Circle you Inſcribe an Iſoſceles Triangle as *a b c*, ſo that *a b* be equal to *b c*: I ſay that Triangle ſhall be greater than half the Segment. For if you draw a Tangent *e b d* which ſhall be Parallel to *c a*; and which ſhall be as *c a* is Perpendicular to the Radius *b f*; (4. 5) (4. 6.) And then Compleat the Rectangle *a e d c*; That Rectangle will be greater than the whole Segment *a b c*: But the Triangle *a b c* is the half of that Parallelogram (3. 18) And therefore muſt be greater than half the Segment *a b c*.

29. Let there be a Tangent *a d b*, a Secant *f c b*, a Chord *a c*, and another Tangent *c d*, I ſay that the Triangle *d b c* is more than half the mixt Triangle *a c b*, comprehended between the Lines *a b*, *b c* and the Ark of the Circle *a c*. For in the Triangle *d b c* the Angle *c* being a Right one (4. 5) the ſide *d b* is longer than *d c* (2. 17) That is, than *d a*; which is equal to *d c* (4. 7) Wherefore the Triangle *b d c* (*having a longer Baſe but the ſame heighth with* a d c) muſt be greater than it; (*as may be collected from* (3. 17.) And therefore it muſt be greater than the half of the whole Triangle *a c b*. But the Triangle *a c b* is greater than the mixt Triangle made by the Ark *a c* and the two Right Lines *a b* and *c b* and therefore the Triangle *b d c* (which is more than the half of *a c b*) muſt be greater than the half of the mixt Triangle *a b c*. Q. E. D.



30. From

30. From these two last Propositions, it follows that by multiplying the sides of Polygons you may make them so *Circumscribed* about or *Inscribed* in Circles, that the Difference by which the Circumscribed exceeds, or the Inscribed wants of the Circle shall be as small as you will. Because if from any Quantity what-ever, you take more than the Half, and from the Remainder more than its half, and again from that Remainder more then its half: You may by doing this very often at last come to leave a Remainder as small as you please; as is Self-evident. Thus (*See the 28th Figure*) After a Triangle is inscribed in a Circle that shall be less than it by three great Segments, you may inscribe an Hexagon that shall exceed the Triangle by those 3 Segments, but shall be less than the Circle by the six little Segments; that are left white in the Figure.

But those six white Segments taken together, do not contain so much space as the half of the three former Shaded ones, (4. 28.) After this you may also Inscribe a *Duodecagon*; which will be lesser than the Circle by 12 smaller Segments; which 12 Segments will still be less than the half of the six Segments of the Hexagon: And thus may you by increasing the Number of sides of the Polygon, Lessen the difference by which the Circumscribing Circle exceeds it, as much as you please. So likewise on the other hand, you might have first *Circumscribed* a Triangle, then an Hexagon, and then a Duodecagon &c. (*and have made*, that way, *the Difference between the Circumscribing Polygon and the Circle, as small as you would.*)

31. Every Circle is equal to a Rectangle Triangle, one of whose Legs is the Radius, and the other a Right Line equal to the Circumference of the Circle. For such a Triangle will be greater than any Polygon *Inscribed*, and less than any Polygon *Circumscribed*, (by 24, 25, 26 and 27 of this 4th Book) And therefore must be equal to the Circle.

For

For should it be greater than the Circle, be the excess as little as it will, a Polygon may be Circumscrib'd, whose difference from the Circle shall be yet *less* than the difference between that Circle and the Rectangl'd Triangle: And that Polygon will be less than the Triangle; which is Absurd. And if it be said that this Rectangl'd Triangle is less than the Circle; an Inscribed Polygon may be made, which shall be greater than that Triangle, which is impossible.

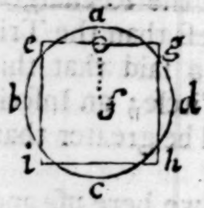
"This kind of Demonstration which we here use and "which is call'd *Reductio ad Absurdum sive ad Impossibile*, "is one of the finest Inventions of the Ancients: And "on it is founded all the *Geometry* of *Indivisibles*; so that "I cannot but much wonder some of our Modern Au-"thors should reject it as indirect and deficient. But if "we must arrive to such a point of Niceness, that we "can't bear any Demonstration, unless it be Direct and "Positive; 'tis easie enough to give this before us such "a turn, as shall render it Regular and Direct.

"For this cannot but be admitted as a Principle; *That* "*if two determinate Quantities* a *and* b *are such, that every* "*other imaginable Quantity, which is greater or less than* a, "*is also greater or less than* b; *Those two Quantities* a *and* "b *must be equal.* And this Principle being granted, "which is in a manner Self-evident, it may directly be "prov'd that the Triangle (before mention'd) is equal "to the Circle: Because every Imaginable Inscrib'd Fi-"gure which is less than the Circle, is also less than the "Triangle: And every Circumscrib'd Figure greater "than the Circle, is also greater than the Triangle."

This is that which is call'd the *Quadrature* of (or *Squaring) the Circle*, which consists in finding a Square, Triangle, or any other Rectilineal Figure exactly equal to a Circle. And this would easily be done, could we find a Right-line equal to the Circumference; as is plain from this last Proposition. But such an Equality is not to be found *Geometrically*.

D 32. If

32. If a Right-line could be dispos'd into the form of the Circumference of a Circle, it would contain more Space than any other Figure, or Regular Polygon whatsoever. Suppose the Circumference of the Circle *a b c d*, to be dispos'd into the Form of a Square, or into any other regular Polygon: So that all the Sides *e g*, *g h*, *h i* and *i e* together may be equal to the Circumference *a b c d*: I say the Circle is greater than that Square. For the Circle is equal to a Rectangle Triangle one of whose Legs is the Radius *f a*, and the other the Circumference. And the Polygon is equal also to such a Triangle, one of whose Legs is the same Circumference *a b c d* or the Sum of the Sides *g e i h*; and the other Leg is the Line *f o* (4. 25.) But as the Line *f o* is less than the Radius *f a*, so the second Triangle which is equal to the Polygon, must be less than the first which is equal to the Circle, and therefore the Square or Polygon must be less than the Circle, which was to be Demonstrated.



" And this is what we mean when we usually say, " that of all *Isoperimetrical Figures* (or which have equal " *Perimeters* or *Circumferences*) the greatest is the Circle.

BOOK

BOOK V.

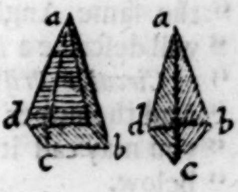
Of Solids.

1. A Right Line is said to be *Right* upon a Plane, when it stands on it at Right Angles, just like a Pillar on the Ground, and is inclined no more to any one side of the Plane, than to the other.

2. Two Planes are parallel to each other, when all the Perpendiculars that can be drawn between them, are equal. (That is, *when they every where are equally distant.*)

3. One Plane is *Right* or perpendicular to another Plane, when like a well-made Wall, it inclines and leans on one side no more than it doth on the other.

4. A *solid Angle* is made by the meeting of three or more Planes, and those Joining in a Point; like the Point of a Diamond well cut.

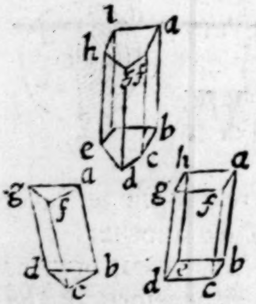


5. If we imagine a Line as *ab*, fixt above in the Point *a*, to be moved along the sides of any Polygon *d b c*; that Line by its Motion shall describe a Figure that is called a *Pyramid*.

6. The *Polygon* is called the *Base* of the Pyramid.

7. If a Line fastned, as before, move round a Circle, as *d b c*, it will describe a Cone; and the Circle is its Base. And

a Line drawn from the Centre *e* to *a*, is call'd its *Axis*.



8. If a Line *a b* move uniformly about two Polygons *g f a* and *d c b*, which are every way equal, having their Sides and Angles mutually Parallel and Corresponding exactly to one another, as *a f* to *b c*, *f g* to *d c*, &c. then that Line shall by its Motion describe a Figure which call'd a *Prism*, and the Polygon is its Base.

9. If the Base of a Prism be a *Parallelogram*, then that *Prism* is call'd a *Parallelopiped*.

10. If a Line *a b* move uniformly round two equal and Parallel Circles, it shall describe or generate a *Cylinder*.

11. The Line joyning the Centres *e*, *e*, in the two Bases is call'd the *Axis*.

" There is no need of conceiving two Bases, equal, " Parallel and opposite, for the Genesis of Prisms and " Cylinders. For they will be describ'd as well by Ima- " gining a Line moving round the Circumference of " any plane Figure with a Motion always Parallel to its " self in its first Position. As if *a b* be suppos'd to be " carried round any of the Bases *d c b*, keeping always " the same Angle with the Plane which it first had, it " will describe a *Triangular*, *Quadrangular*, *Quinquangular*, " or *Circular Prism* according to the Figure of the Base. " And the upper End of the Line will describe a Base, (as " you may call it) at the Top, equal and Parallel to that " below.

12. All these Figures, if the Axis be *Perpendicular* to the Base, are call'd *Isosceles* Prisms, or Cylinders; But if the Axis be any way *inclin'd*, they are call'd *Scalenes*.

13. If

13. If a Semicircle *a d b* be turned quite round on its Diameter *a b*, it will describe a *Sphere* or *Globe* whose Axis will be *a b*, and its Centre *c* the same with the Semicircle. Every Line passing thro' the Centre *c* and terminated at each end by the Surface of the Sphere, is call'd a Diameter, and may be call'd an Axis.

14. All Lines drawn from the Centre *c* to the Surface, are call'd Radius, and are all equal to one another.

15. Two Right-lines if they meet so as to cut or cross each other, are in the same Plane: Wherefore all the Angles and Sides of every Triangle are in the same Plane.

16. If two Planes *e b d* and *a g f* cut or Intersect one another, they shall do so in a Right-Line, as *b d*; which is call'd their common Section.

17. If a Right-line *c d*, be Perpendicular to two Lines *d f* and *d g* which are in the same Plane, that Line is also Perpendicular to that Plane.

18. If a Right-line *d c* be Perpendicular to three Right-lines *d f*, *d g* and *d a*, they are all three in the same Plane.

19. If two Lines *d c*, *b i* are Perpendicular to the same Plane *f g a* they will be Parallel to one another.

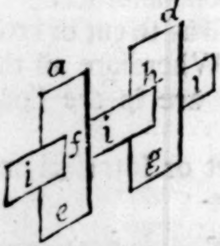
20. If two Lines *d c*, *b i* are Parallel; and you draw another Line, from any Point in one, to the other, as *b d*; those three will be all in the same Plane.

21. If two Lines *d c*, *b i* are Parallel to a third *a k*, tho' that third Line be not in the same Plane with them, yet they shall be Parallel to each other.

22. If a Line *a b* be Perpendicular to, (*or make any other equal Angles with*) two Planes *f e* and *c d*, those Planes are Parallel.

23. If two Parallel Planes, *d h g* and *a f e* are cut by a third *i i i*, the common Sections *f e* and *h g* are Parallel.

24. If a Solid Angle be made by three Plane Angles, any two of those are always greater than the third.



All these Propositions are so manifest to one that will but consider them with a little Attention, that 'tis needless to stay to Demonstrate them. (And indeed the solemn and Regular Demonstration of a thing Plain in it's self, always makes it more Obscure.)

25. The Plane Angles concurring to make a Solid

28. Every Parallelopiped is divided into two equal Triangular Prisms, by a Diagonal Plane, which is Perpendicular to its Base.

29. Triangular Prisms having equal Bases (and *Heights*) or being between the same Parallels are equal.

30. Pyramids having equal Bases and Heights are also equal.

31. All Prisms in General, all Cylinders and Cones, with equal Bases and Heights, are equal.

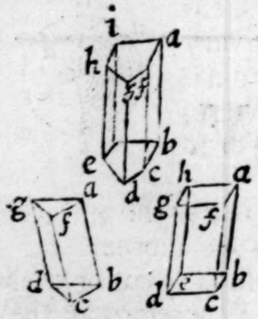
32. Pyramids and Cones on equal Bases and of equal Heights with Prisms and Cylinders, are one third of such Prisms and Cylinders.

33. Every Sphere is equal to a Cone whose Perpendicular Axis is the Radius of the Sphere, and its Base a Plane equal to all the Convex Surface of it.

34. Of all Solid Figures that can be encompass'd or terminated by the same Surface, the greatest is a Spherical one.

35. That is call'd a *Regular Body*, whose Surface is compos'd of Regular and equal Figures. And whose

a Line drawn from the Centre *e* to *a*, is call'd its *Axis*.



8. If a Line *a b* move uniformly about two Polygons *g f a* and *d c b*, which are every way equal, having their Sides and Angles mutually Parallel and Corresponding exactly to one another, as *a f* to *b c*, *f g* to *d c*, &c. then that Line shall by its Motion describe a Figure which call'd a *Prism*, and the Polygon is its Base.

9. If the Base of a Prism be a *Parallelogram*, then that *Prism* is call'd a *Parallelopiped*.

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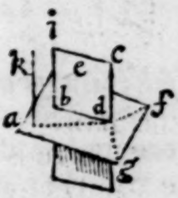
16. If two Planes *e b d* and *a g f* cut or Intersect one another, they shall do so in a Right-Line, as *b d*; which is call'd their common Section.



17. If a Right-line *c d*, be Perpendicular to two Lines *d f* and *d g* which are in the same Plane, that Line is also Perpendicular to that Plane.

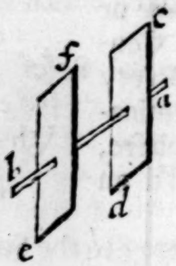
18. If a Right-line *d c* be Perpendicular to three Right-lines *d f*, *d g* and *d a*, they are all three in the same Plane.

19. If two Lines *d c*, *b i* are Perpendicular to the same Plane *f g a* they will be Parallel to one another.

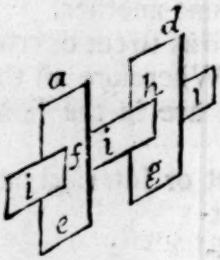
20. If two Lines *d c*, *b i* are Parallel; and you draw another Line, from any Point in one, to the other, as *b d*; those three will be all in the same Plane.

21. If two Lines *d c*, *b i* are Parallel to a third *a k*, tho' that third Line be not in the same Plane with them, yet they shall be Parallel to each other.

22. If a Line *a b* be Perpendicular to, (*or make any other equal Angles with*) two Planes *f e* and *c d*, those Planes are Parallel.



23. If two Parallel Planes, *d h g* and *a f e* are cut by a third *i i i*, the common Sections *f e* and *h g* are Parallel.



24. If a Solid Angle be made by three Plane Angles, any two of those are always greater than the third.

All these Propositions are so manifest to one that will but consider them with a little Attention, that 'tis needless to stay to Demonstrate them. (*And indeed the solemn and Regular Demonstration of a thing Plain in it's self, always makes it more Obscure.*)

25. The *Plane Angles*, concurring to make a *Solid* one, taken all together, are always less than four Right ones. For if they should make four Right Angles, they would form a Plane and not an Angle. Wherefore that they may make a Solid Angle they must be less than four Right ones.

Tis a very good way in order to gain a clear Idea of Solids and their Angles, to make the Regular Bodies out of thick Paper or Past-board, as you are directed by Dr. Barrow, *at the End of his thirteenth Book of* Euclide; *and by many other Authors.*

26. In all Parallelopipeds the opposit Planes are equal; as is easie to conceive (from 5. 9.)

The eight following Propositions (*says our Author*) *are Demonstrated in the second Part of the Elements: And might be so here, by applying to Solids what is* Demonstrated *of Planes in the third and fourth Book. But there is no need to stay to do it now.*

27. All Parallelopipeds having equal Bases (and Heights) or being between the same Parallels are equal (3. 14.)

28. Every

28. Every Parallelopiped is divided into two equal Triangular Prisms, by a Diagonal Plane, which is Perpendicular to its Base.

29. Triangular Prisms having equal Bases (and *Heights*) or being between the same Parallels are equal.

30. Pyramids having equal Bases and Heights are also equal.

31. All Prisms in General, all Cylinders and Cones, with equal Bases and Heights, are equal.

32. Pyramids and Cones on equal Bases and of equal Heights with Prisms and Cylinders, are one third of such Prisms and Cylinders.

33. Every Sphere is equal to a Cone whose Perpendicular Axis is the Radius of the Sphere, and its Base a Plane equal to all the Convex Surface of it.

34. Of all Solid Figures that can be encompass'd or terminated by the same Surface, the greatest is a Spherical one.

35. That is call'd a *Regular Body*, whose Surface is composed of Regular and equal Figures. And whose Solid Angles are all equal, as are ——

36. The *Tetrahedron*, which is a Pyramid, comprehended under 4 equal and equilateral Triangles; so that its Base is equal to each Side.

37. The *Hexahedron* or *Cube*, whose Surface is compos'd of six equal Squares, like Dice which are us'd in Play.

38. The *Octahedron*, which is bounded by eight equal and equilateral Triangles.

39. The *Dodecahedron*, which is contain'd under twelve equal and equilateral Pentagons.

40. The *Icosihedron*, consisting of twenty equal and equilateral Triangles.

41. Besides these five *Regular Bodies*, 'tis not possible to find any others that shall correspond to the Definition; which is thus Demonstrated.

To begin with equilateral Triangles, which are the most simple of all Rectilineal Figures. Of these there must be three at the least to make a Solid Angle, and

three of them join'd together will juſt make the *Tetrahedron*. For thoſe three Triangles meeting in a Point do form a Triangular Baſe ſimilar and equal to the Sides; as appears by the bare compoſition of the Figure, four ſuch Triangles join'd together in a Point make the Angle of the *Octahedron*.

By joining five ſuch Triangles together the Angle of the *Icoſihedron* is form'd.

But ſix ſuch Triangles join'd in a Point cannot make a Solid Angle: Becauſe they make four Right ones (*for every Angle of an equilateral Triangle is* $\frac{1}{3}$ *of two*, *or* $\frac{2}{3}$ *of one Right Angle, either of which fractions Multiplied by ſix, gives four Right Angles.*) Whereas every Solid Angle is made up of ſuch plane Angles as all together muſt be leſs than four Right ones (5. 25.) So that with Triangles 'tis Impoſſible to form any more Regular Bodies than theſe three.

Next if you take *Squares* and join three of them together, they will make the Angle of the Cube: And there can no other regular Body but a Cube be made with Squares, for four Squares join'd together will not make a Solid Angle but a Plane. (5. 25.)

If you join the Angles of three Pentagons together you will conſtitute the Angle of the Dodecahedron: But four ſuch Angles cannot make a Solid one.

And laſtly, *Three Hexagons* joined together do make juſt four Right Angles, and therefore they cannot make a Solid Angle: And as for three *Heptagons*, or other Figures of yet more Sides, they can much leſs do it; (*becauſe their Angles being very obtuſe, three of them will exceed four Right ones.*) So that upon the whole 'tis plain, that of theſe five *Regular Bodies*, three are made of Triangles, one of Squares, and one of Pentagons, and *there can be no other.*

BOOK

BOOK VI.

Of Proportion.

1. WHEN we speak of *Magnitude* and say that any Quantity is *great*, we always make a Comparison between that Quantity and some other of the same Nature, in respect to which we say that it is *great*.

Thus we say of an *Hill* that 'tis *Little*, or of a *Diamond* that 'tis *Large*, because we compare that Hill with others that are Higher, and in respect of them 'tis Little; and we compare that Diamond with others that are Little, and in respect of them we say 'tis a Large one.

2. When we consider one Quantity in respect of another, to see what Magnitude it hath in comparison of that other: That Magnitude so found is call'd its *Ratio*, or *Reason*; tho' it would be more intelligible if it were call'd *Comparison*.

3. That Quantity which is compar'd with another is call'd the *Antecedent* and *that other* with which it is compar'd, is call'd the Consequent.

4. When we consider four Quantities and compare them (*by Pairs*) two with two; as *a* four with *b* 2, and *c* 6 with *d* 3. If we find that *a* hath as much Magnitude (or is as big) in comparison of *b*; as *c* hath in comparison of *d*: Then we say that their *Ratio*'s are Equal; that is, the *Ratio* that *a* hath to *b* is equal to the *Ratio*

of

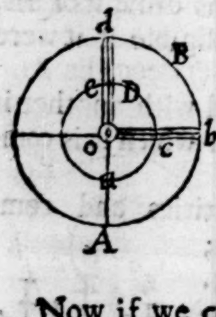
of *c* to *d*: For as *a* is twice as big as *b*; so *c* is twice as as big *d*.

5. But if *a* 4 hath more Magnitude in respect of *b* 2, than *c* 6 hath in respect of *e* 5. That is, if as *a* 4 is twice as big as *b* 2, *c* 6 be found not to be twice as big as *e* 5. Then the *Ratio*'s are unequal: And we say *a* hath a greater *Ratio* to *b* then *c* hath to *e*. So that to have a *greater Ratio*, is nothing but to have more Magnitude, or to be bigger, in respect of a second Term, than a third is in respect of a fourth.

6. The Equality of *Ratio*'s is call'd *Proportion*; and when we find that of four Quantities or Numbers, the first hath as much Magnitude (*or is as big*) in respect of the second; as the third is in respect of the fourth; then we say that those four Quantities are *Proportionals*.

The better to make the Mysteries of Proportion comprehended, which pass for the most difficult things in Geometry, *as unquestionably they are the most Important, I will explain them by an Example; which (in my Opinion) will render all those things very Intelligible, which otherwise appear very perplext.*

7. Let us imagine the Circle *b A d* to be describ'd by the Motion of the Line *o b* round the Centre *o*. And at the same time let the Circle *c a e* be describ'd by the Motion of a Point *c* in the Line *o b*. Let us suppose also that the Line *o b* be moved once round again, and at last to stand in the Position *o d*. Let the Ark *d B b* be call'd *B*, and the Ark *e D c*, be call'd *D*. Let *A* be put for the whole outer Circle, and *a* for the whole inner one.



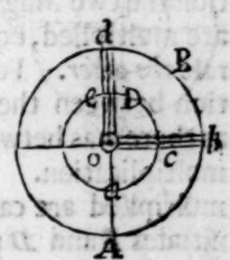
Now if we compare the whole Circle *A* with its Ark *B*, and the whole other Circle *a* with its Ark *D*. We shall find plainly, that the Circle *A* is just as big in respect of the Ark *B*, as the inner one *a* is in respect of the Ark *D*; and therefore if *B* be a fourth, or any other part

part of the Circle *A*, *D* alſo will be a fourth, or the ſame proportionable part of its Circle *a*. Which we uſually expreſs by ſaying, as *A* is to *B*, ſo is *a* to *D*. And write it thus *A*. *B* : : *a*. *D*.

8. If you ſhould change the Order of the Terms, and compare *B* with *A* and *D* with *a*; you will find plainly that *B*. *A* : : *D*. *a*. So that ſuppoſing *A*. *B* : : *a*. *D*, we cannot but preſently conclude by *Inverſe* Proportion, that *B*. *A* : : *D*. *a*.

9. If you change them ſo as to compare Antecedent with Antecedent and Conſequent with Conſequent; you will find *Alternately*, that *A*. *a* : : *B*. *D*. And this is very plain; for if the whole Circle *A* be double, Triple of, or in any other Proportion, to the Circle *a*; the Ark *B* muſt be alſo double, triple of, or in the ſame Proportion to the Ark *D*. This I ſay is plain, becauſe the two Circle *A* and *a* are deſcrib'd by the Motion of the Line *o c b*; ſo that while *b* deſcribes the Circle *A*, *c* deſcribes the inner Circle *a*: and while *b* deſcribes the Ark *B*; *c* alſo deſcribes the Ark *D*. And this by one common circular Motion; only the Point *c* moving much ſlower than the Point *b*, deſcribes a Circle much leſs, in proportion to the ſlowneſs of its Motion: Thus alſo when the Point *b* ſhall have deſcrib'd the Ark *B*, the Point *c* in like manner will have deſcrib'd the Ark *D*, which will be much leſs than *B*; in proportion to the ſlowneſs of its Motion.

10. If we compare the differences between the Antecedents and Conſequents, with their Conſequents; as for Inſtance, *A* leſs *B* with *B*, and *a* leſs *D* with *D*, we ſhall find they alſo are Proportional: And that *A* leſs *B*. *B* : : *a* leſs *D*. *D*.



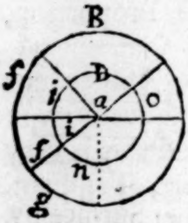
For 'tis manifeſt that the Ark *b A d* (which is *A* leſs *B*) is to *B* as the Ark *c a e* (which is *a* leſs *D*) is to *D*. And this is call'd *Proportion by Diviſion*.

11. I

11. If we add the Antecedents and Consequents together; we shall find that *A* more *B.* is to *B*:: as *a* more *D* is to *D*. Which is call'd *Composition.*

12. And if we should say that *A. A* less *B*:: *a. a* less *D.* This kind of Proportion is call'd *Conversion.*

13. If we take many Quantities which are proportionable and consider them in Rows two by two: As *B.f*:: *D.j.* and *f.g*:: *i.n*, &c. Then we may conclude that if we compare the first and the last; *B.g*:: *D n.*



And this is call'd Proportion by Equality: in *Latin*, *Ex æquo ordinata.*

The Proporsition which follows is a little Intricate, but of no great Importance, so it may be omitted.

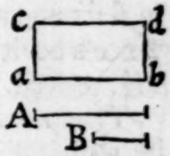
14. But if you take *f.g*:: *o.D.* That is, as the last save one, is to the last of all in the first Row, so is some other Quantity *o* to the first in the second Row; then we conclude that *B.g*:: *o.i.* That is; as the first is to the last in the first Rank:: so is this other Quantity *o* to the last save one in the second Rank. And this Proportion is call'd *ex æquo Perturbata.* But it may always be justly infer'd, for since *f.g*:: (or *i.n*::) *o.D.* Alternately and Inversly it will follow that *o.i*:: *D.n.* or as *B.g.*

15. If *B* be taken as often as *D. ex. gr.* 3 *B* and 3 *D.* we may conclude that *B.D*:: 3 *B.* 3 *D.* or as 10 *B* to 10 *D*; or also as 12 $\frac{1}{2}$ *B.* to 12 $\frac{1}{2}$ *D.* And so on in whatsoever Proportion the two Magnitudes *B* and *D* are multiplied, so they are multiplied, equally, *or that you take one as often as you take the other.* For then there will be the same Proportion between the Magnitudes thus equally multiplied, as there was between the simple Magnitudes, before such multiplication. And these Magnitudes thus equally multiplied are call'd *Equi multiples* of the simple Magnitudes *B* and *D*; and we say that *Equi multiples* are in the same Proportion as such simple Magnitudes, *out of which they are compounded.*

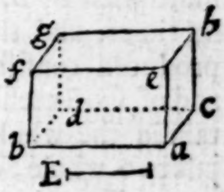
16. If

16. If *B* be divided in the same manner as *D* is; and *ex. gr.* you take a fourth part of *B* and the like of *D*; or the tenth or any other part of *B*, and the same of *D*. Then will these Parts be Proportional to their wholes, $B . D :: \frac{1}{4} B$ (or $\frac{1}{10} B$.) is to $\frac{1}{4}$, or $\frac{1}{10} D$. All which is Self-evident.

17. To multiply one Line by another is to make a Rectangl'd Parallelogram, whose two Contiguous Sides shall be the two Lines given. Thus if you multiply the Line *A* by *B*; 'tis the same thing as to make the Rectangle *abcd*; whose Side *ab* is equal to *A* and *ac* to *B*.

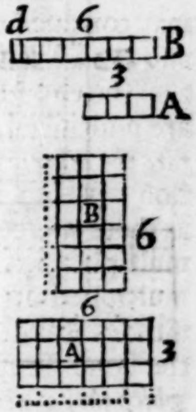


18. To multiply a Rectangle or any other Surface by a Right-line, is to make a Rectangle Parallelopiped (*or Prism*) (5. 9.) whose Base shall be the Surface given, and its Perpendicular height the Line given.



Thus to multiply the Surface *abdc* by the Line *E*, is the same thing as to make a Solid *abfghe* whose Base is the Surface given *ad*, and its height *ae* or *bf*, equal to *E* the Line given.

19. And the better to conceive these Multiplications, we may imagine the two Lines as if they had some little breadth, and their whole lengths to be divided into little Squares as you see in these Figures: Where *A* is a Line (or rather a Ruler) containing 3 small Squares; and *B* is another Ruler containing six small Squares of the same breadth with those in *A*.



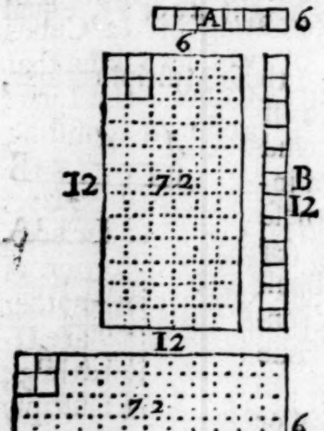
Wherefore to multiply *B* by *A* or *A* by *B*, is to take the Rule *B*, as often as there are little Squares in *A*, or which is all one at last, to take *A* as often as there are Squares in *B*;

ſo *B*, being taken three times makes the firſt Rectangle, containing 18 ſuch little Squares; and *A* taken 6 times will make another Rectangle containing alſo 18 Squares and in the whole Equal to the former.

20. Obſerve here, that the ſame Multiplication will be made tho' the length of either Line doth not conſiſt exactly of ſuch a certain Number of ſmall Squares. As for inſtance ſuppoſe in *A* there be juſt three Squares, but in *B* ſix and an half, a Quarter or any other Part: Or any Exceſs be it what it will, and there mark'd with *d* (ſee *Fig.* preced.)

Then need *B* be taken only three times (as before) to be multiplied by *A*, and the Product will be the firſt Rectangle *B* containing 18 whole Squares and ſix of the Parts or Exceſſes *d*. So in like manner if *A* had been multiplied by *B*, that is taken ſix times and an half, or ſix times, and the exceſs *d*; The Rectangle *A* would be produced conſiſting as the former of 18 whole Squares and of ſix Exceſſes of *d*. *That is each Rectangle will* contain in the whole (if *d* be half of one of the Squares) juſt two whole Squares.

21. If we imagine the former Line *B* to be contracted one half; (*Saith* Pardie, *tho' what he means by it is not eaſie to conceive, and his* Latin *Tranſlator leaves it as he found it)* Or *if we imagine the little Squares in* B *to be ſubdivided into others, that are juſt half of the former*; after this, its length remaining ſtill the ſame, it will have 12 ſmall Squares (*i. e.* its length will be 12 times greater than its Latitude) and if *A* be contracted as to Latitude (*or have its Diviſions or Squares leſſen'd in the ſame Proportion*:) Then in *A* there will be ſix ſmall Squares: So that now if *A* be multiplied by *B*, or *B* by *A*:



A: Two Rectangles will be produced equal in the whole to the two former, for *B* taken six times makes the first Rectangle containing 72 small Squares; and *A* taken 12 times makes the other Rectangle containing also 72 Squares: Now these 72 Squares are in reality neither more nor less than the 18, of the other Rectangles, (*tho' drawn larger that the Subdivisions may appear*) because one of those 18 contain'd as much space as, or answer'd to 4 of these 72, as is apparent from the (*Black Lines in the*) Figure: So that whatsoever breadth, tho' never so small be given to these Lines, by Contracting (*or subdividing the little Squares*) infinitely, 'tis plain the Rectangles made out of them by Multiplication, will still be the same. And from hence we may boldly suppose the Lines, for Indivisibles; and that they may be multiplied one into another, when a Rectangle is made out of them; because the Magnitude of this Rectangle never varies, tho' some very small Breadth be allow'd to the Lines.

22. 'Tis very easie to apply all this to the Multiplication of Solids; But then instead of Squares, we must Imagine Cubes: For if you conceive a Surface compos'd of 12 Cubes, and on the other side a Line consisting of two Cubes: The Superficies consisting of 12 Cubes will be multiplied by the Line of two, by taking that Superficies as often as there are little Cubes in the Line; *i. e.* Twice, and so a Solid will be produced consisting of 24 small Cubes.

23. By all which it appears, that these small Squares and Cubes in the Multiplication of Lines and Surfaces, are the same things as Unites in the Multiplication of Numbers. For as to multiply one Number by another *v. gr.* 3 by 5, is only to take 3 as often as there are Unites in 5, or to take 5 as often as there are Unites in 3; and either way the Product will be 15: So to multiply one Line by another is to take either of those Lines, as often as there are little Squares in the other: And to multiply a Surface by a Line, is to take that Surface, as often as there are (*supposed*) Cubes in that Line.

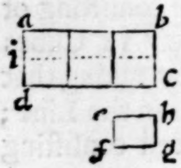
We

We shall speak in another Place of the Multiplication of Surfaces by Surfaces, or by Solids: From whence result such Compositions, as are said to have more than three dimensions.

24. All Magnitudes may be express'd by Lines: As if one Magnitude be double or triple of another, or in any other *Ratio*, two Lines may easily be taken, of which one shall be double or triple of the other, or in any other like Proportion with those Magnitudes: So for Instance, to express two Times, as one Hour and two Hours; or two Velocities of which one shall be double to the other; you need only take two Lines, as *a* double of *b*; and then you may say that *a* represents two Hours or Velocities, and *b* answers to one of each: and then you may proceed to compute with those two Lines, as with the Hours and Velocities themselves, &c.

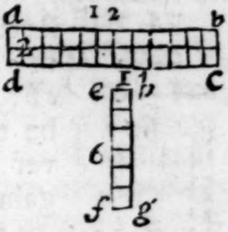
25. To know the Proportion of Rectangles, the *Ratio* of the Length of One to the Length of the Other, and moreover the *Ratio* of the Breadth of One to the Breadth of the Other must be known.

For *Example*; To know what Proportion the Rectangle *ac* hath to *eg*: 'Tis not enough only to know that the Length *ab* is triple of *eb*; but it must be known also that *ad* is double of *ef*. For if *ai* be taken equal to *ef*, the Rectangle *bi* will be triple of *eg*, because *ab* is triple of *eb*, and *ai* equal to *ef*. And moreover because *id* is also equal to *ai*, or *ef* (for *ad* is supposed to be double of *ai*, and of *ef*) the Rectangle *ic* shall also be triple of *eg*; so that the whole Rectangle *ac* is twice triple of the Rectangle *eg*; that is sextuple of it, or containing it six times. And what we say now only of the double or triple *Ratio* of their Breadths and Lengths, is also to be understood of any other *Ratio*, be it what it will: For if *ab* be quadruple of *eb*, and *ad* triple of *ef*, the Rectangle *ac*, will be three times quadruple of the Rectangle *eg*: that is duodecuple of it, or doth contain it twelve times.



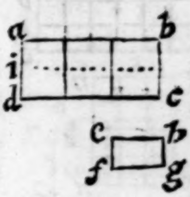
But

But if *a b* be duodecuple of *e b*, and at the same time *e f* be Triple of *a d*, then there is a certain Compensation made: For if Respect were had to their Breadths *a b* and *e b* only, the Rectangle *a c* would exceed the other, nay indeed contain it 12 times: Neverthelefs this excefs is loft *(in fome Meafure)* in refpect of their Altitudes or Heights *a d* and *e b* which if only confider'd, the Rectangle *eg* would be Triple of *a c*.



But then when we come to compare thefe feveral Exceffes and Deficiencies together; we fhall find that the Rectangle *a c* being one way 12 times greater and the other way 3 times lefs than *e g*, will be at laft but only four times as great.

26. And this is what we mean, when we fay, that all Rectangles are to each other in a *Ratio Compounded* of that of their fides; for if *a b* be triple of *e b* and *a d* double of *e f*, the Rectangle *a c*, fhall be to the Rectangle *e g* in a *Ratio* compounded of the triple and the double, that is, it fhall be thrice double, or twice triple, or in one Word Sextuple. So alfo if *a b* were quadruple of *e b* and *a d* triple of *e f*; the Rectangle *a c* would then be to *e g* in a *Ratio* compounded of the Quadruple and the Triple; fo that it would have been three times Quadruple, or 4 times Triple, or in one word Duodecuple of *e g*.

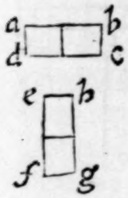


Moreover, if *a b* were Duodecuple of *e b*, and *a d* Subtriple of *e f*, (that is, if *e f* be Triple of *a d*) the *Ratio* of the Rectangle *a c* to *e g* would be compounded of the duodecuple and fubtriple *Ratio*; fo that *a c*, would have been 12 times fubtriple of, or in one word Quadruple of *e g*.

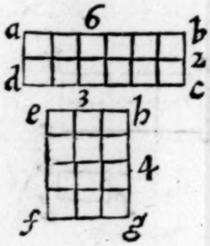
If you take the third part of a Crown 12 *times it will make, or be equal to four whole Crowns: So that four*

Crowns are 12 *times subtriple of one Crown; that is, do make* 12 *thirds of a Crown.*

27. From whence it will appear that if the sides of two Rectangles are reciprocally Proportional, those two Rectangles are equal: For if *a b* be double to *e h*, and reciprocally *h g* be double to *c b*: Or if *a b* be triple of *e h*, and then *h g* be triple of *b c*; or in a word, if whatever *Ratio a b* hath to *e h*, *h g* hath back again the same *Ratio* to *b c*, 'tis plain that as much as the first Rectangle *a c* exceeds the other in length, just so much is it exceeded by the other in breadth; so that the length of one Compensates for the breadth of the other, and consequently it must be equal. And from hence is deduced this most useful and important Proposition: That,



28. If four Quantities (*or Numbers*) be proportional, the Product arising from the Multiplication of the two middle Terms, is always equal to that, which is made by the Multiplication of the two Extreams. As if *a b. e h* :: *h g. b c.*

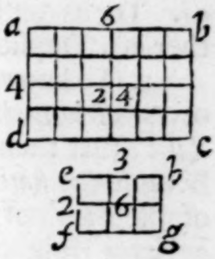


I say, from the Multiplication of the extreams *a b* by *b c* there is produced the Rectangle *a c*: And by multiplying the middle Terms *e h* and *h g*, there is produced the Rectangle *e g*; and those two Rectangles *a c* and *e g* are equal. (6. 27.) (*Because, as much longer as* a b *is than* e h, *just so much longer is* h g *than* b c.)

What is thus done by Lines and Rectangles, may be done by any Quantity whatsoever; because all Quantities can be express'd by Lines, and all Multiplications of Magnitudes by Multiplications of Lines, *i. e.* by Rectangles. (6. 24.)

29. When Rectangles have their sides Proportional, so that *a b. e h* :: *a d. e f.* then is the Rectangle *a c* to the Rectangle *e g*, in a *Duplicate Ratio*, to that of their Sides: For the *Ratio* of *a c* to *e g*, is Compounded of the *Ratio* of

of *ab* to *eh*, and of the *Ratio* of *ad* to *ef*. (6. 26.) But the *Ratio* of *ab* to *eh* is in this Case, (by the Supposition) the same as the *Ratio* of *ad* to *ef*; so that to gain the *Ratio* which the Rectangle *ac* hath to *eg*. We need only take *twice the Ratio* of *ab* to *eh*. For Example, if as here *ab* be double to *eh*, and *ad* double to *ef*, the Rectangle *ac* shall be twice double, that is, Quadruple of the Rectangle *eg*. And if *ab* had been triple of *eh*, and consequently *ad* triple of *ed*: Then the Rectangle *ac* would have been three times triple, that is nine times as big as *eg*: Or if *ab* had been Quadruple of *eh*, *ac* would have been 16 times as great as *eg*.



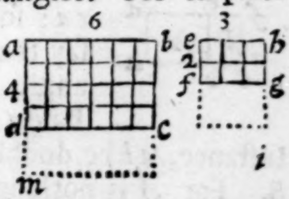
30. If a third Line be taken as *no*; and it be so proportional that *ab*.*eh*::*eh*.*no*. Then shall the two Rectangles *ab* and *eg* be to one another as the two Lines *ab* and *no*. (vid. *Fig. Preced.*)

n |.....| *o*

For *ab* is to *no* in a duplicate *Ratio* of *ab* to *eh*. And if *ab* had been, (as it is double,) triple or quadruple of *eh*. Then would *ab* have been in a *Ratio* three times triple; or four times quadruple of (*that is nine or* 16 *times as great as*) the *third Proportional* no.

31. Those Rectangles which have their sides thus Proportional: That *ab*. *eh*::*ad*. *ef*. are call'd *Similar*, whose *Homologous Sides* are those which answer each to other in the Proportion, as *ab* and *eh*, or *ad* and *ef*: For as *ab* is the greatest side of the Rectangle *ac*, so *eh* is also the greatest side of the Rectangle *eg*.

32. All Squares are similar Rectangles. For 'tis plain that if *ab* be double or triple of *eh*, *am* must also be double or triple of *hi*: Because *am* is equal to *ab*, and *hi* to *eh*.



33. All similar Rectangles are to each other as the Squares

 of

of their Homologous Sides. I say the Rectangle *a c* is to the Rectangle *e g* :: as the Square *b m* to the Square *e i*. For as well Squares as Rectangles are to one another in a Duplicate *Ratio* of *a b* to *e b* (6. 29, 30.)

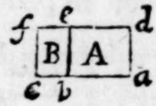
34. To know the *Ratio* between two solid Rectangles, or *Parallelopipeds* there ought to be known the several *Ratio's* that their Bases and Heights have to each other; Because the *Ratio* of one Solid to another is compounded of the *Ratio's* of their Lengths and Breadths and Thicknesses or Heights; as is easie to conceive, if that be well understood which hath been said, about the Proportions of Rectangles. For if one Parallelopiped hath its Base double to the Base of another, and its Height triple of the Height of the other: The former will be twice triple, or three times double, or in one word Sextuple of the latter.

35. If the *Bases* of two Parallelopipeds be *Reciprocally* as their *Heights*, those Parallelopipeds are equal: Which is prov'd as the 27th of this Book; for as much as one exceeds the other in Breadth and Length, so much doth the other exceed it in Height.

36. When Parallelopipeds have all their sides Proportional, they are call'd *Similar*; and they are in a *Triplicate Ratio* of their Sides, as it hath been prov'd of Rectangles, that they are in a *Duplicate Ratio* of their side.

37. Similar Parallelopipeds are to one another as the Cubes of their Homologous Sides; for both Cubes and Parallelopipeds are in a *Triplicate Ratio* of their Homologous Sides.

38. All Rectangles having the same Heights are to one another as their Bases.

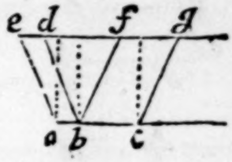


Let the Rectangles *A* and *B* be between the same Parallel Lines *d f* and *c a*; so that *a d* be equal to *c f*; than do I say that *A. B* :: *a b. b c*. That the Rectangle *A.* is to the Rectangle *B.* as the Base *a b.* to the Base *b c*: And that if, for Instance, *a b* be double to *b c*, then shall *A* be double to *B*. For *A* is nothing but the Line *b a* multiplied by

d a

d a (6. 17.) and *B* is nothing but the Line *c b* multiplied by the same Line *a d* or which is all one) *b e* or *f c*. Wherefore (6. 15.) *A. B* :: *a b. b c*.

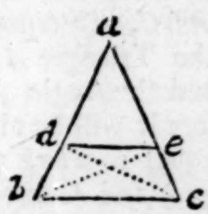
39. All *Parallelograms* which are between the same Parallels (*or which have the same Height*) are as their Bases. I say the Parallelogram *e b* is to the Parallelogram *b g* :: as the Base *a b* is to the Base *b c*. For having made the two Prick'd Rectangles on the same Bases, those will be equal to the Parallelograms, (by 3. 14.) But those Rectangles are as their Bases, (by the Precedent) Wherefore the Parallelograms must also be as their Bases; That is *e b*. *b g* :: *a b. a c*.



40. All Triangles which (*have the same Height*) or are between the same Parallels, are as their Bases; For they are the halves of Parallelograms. (3. 8.)

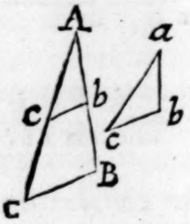
41. When Triangles (as those in the following Figure) have their Bases on one and the same Line, and their *Vertices* or Tops meeting in the same Point; they are then taken to be between the same Parallels, as *a d e* and *c d e*, or *a d e* and *b d e* (*because they have the same Perpendicular Height.*)

42. If in any Triangle a Line be drawn Parallel to the Base, that Line shall cut the Legs Proportionally. Let the Triangle be *a b c* and let the Line *d e* be Parallel to *b c*. I say that *a d. a e* :: *a b. a c* :: *d b. e c*, &c. Draw the Lines *d c* and *e b*, then shall the Triangle *c e d* be to *e a d*, as the Base *e c* is to *a e*, (6. 40, 41.) So also will the Triangle *d e b* be to *d e a* :: as the Base *d b*. is to *d a*. But the Triangle *e c d* is equal to *d e b* (3. 16.) wherefore the Triangle *b d e* (or *c e d*) is to the Triangle *e a d* :: as *b d*. is to *d a* :: or as *c e* to *e a*. Therefore also must *b d. d a* :: *c e, e a* because both the *Ratio* of *b d* to *d a*, and also that



of *ec. ea.* are the very ſame with that of the Triangle *bed* or *ced*, to the Triangle *ade*.

43. If in a Triangle as *acb* you draw a Line *de* Parallel to the Baſe *cb*, I ſay that *ed. cb* :: *ae. ac* :: or as *ad. ab.* For drawing *ef* Parallel to *ab*; *fb* will be equal to *ed* (3. 9.) But by the Precedent *fb. cb* :: *ae. ac.* wherefore *ed.* (*fb*) *cb* :: *ae. ac.* or as *ad.* to *ab.*



44. Thoſe Triangles are call'd *Like*, or *Similar* which have all their three Angles reſpectively equal one to another, *or which are Equiangular*: v. gr. If the Angle *A* be equal to *a* the Angle *B* to *b* and *C* to *c*, then the whole Triangle *ABC* is *Like* or *Similar* to the Triangle *abc*.

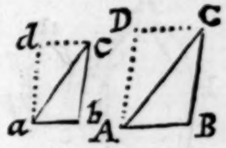
45. If two Triangles have two Angles in one equal to two in the other, the third Angle in one muſt be equal to the third in the other; and the Triangles will be Similar: For ſince the Sum of all the three Angles in each, is equal to two Right ones. (2. 9.) If two Angles in one, be equal to two in the other, the remaining third Angles will be equal.

46. All ſimilar Triangles have their Sides about the equal Angles Proportional. I ſay, *AB. ab* :: *AC. ac* :: *BC. bc*, &c. For take, in the greater Triangle *ABC*, *Ab* equal to *ab*, and *Ac* equal to *ac*; then will the Triangle *Abc* be every way equal to *abc* (2. 11.) and the Angle *Abc* is equal to the Angle *abc*: Wherefore it will be alſo to *B*, which by the ſuppoſition was equal to *abc*, and therefore *cb* is Parallel to *CB* (1. 31.) and Conſequently (by 6. 42, 43.) *Ab. AB* :: *Ac. AC* :: *cb. CB.*

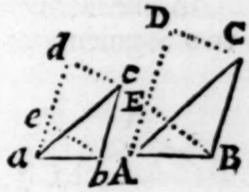
47. All ſimilar Triangles are in a duplicate *Ratio* of, or as the Squares of their Homologous Sides.

Let *abc* be ſimilar to *ACB*; ſo that *ab. AB* :: *bc. BC.* Then firſt, if the Angles *B* and *b* are Right ones, compleat the Rectangles *db* and *DB*; which Rectangles will

will be to one another in a duplicate *Ratio* of the side *b c* to the Homologous Side *B C*, or as *b c* Square to *B C* Square (6. 29. 31.) But the Triangle *a b c* is just half the Rectangle *b d*, and the Triangle *A B C* is just half the Rectangle *D B* (3. 8.) wherefore these Triangles must also be to each other in a duplicate *Ratio* of, or as the Squares of their Homologous Sides.

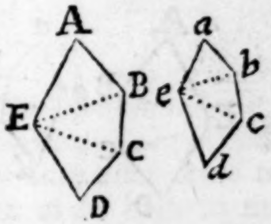


2. If the Triangles are not Rectangular, as in the Figures adjoined, draw the Parallels *a d* and *A D*, and compleat the Rectangles *b e d c* and *B E D C*; then,

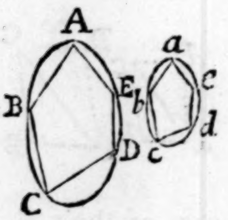


The Triangles *a d c* and *A D C* will be Similar; because the Angle *d* is equal to *D* (as being both Right ones) and also the Angle *d a c* is equal to *D A C* because they are severally equal to the (*Suppos'd*) equal Angles *a c b* and *A C B* (1. 31.) wherefore *a c*. *A C* :: *c d*. *C D* (6. 46.) But *a c*. *A C* :: *b c*. *B C* (by the supposition) and therefore *c d*. *C D* :: *b c*. *B C*. And consequently the Rectangles *b d* and *B* D are Similar (6. 31.) and therefore are as the Squares of their Homologous Sides (6. 33.) And therefore their halves must be so too: *i.e.* the Triangles *a b c* and *A B C* (3. 18.) are to each other in a Duplicate *Ratio* of, or as the Squares of their Homologous Sides. Q. E. D.

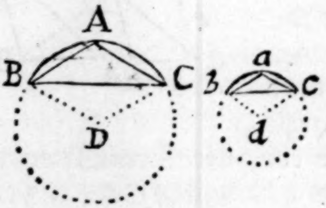
48. Similar Polygons are those which having an equal Number of Sides have all the several Angles in one, equal to those in the other, and also the sides about those equal Angles Proportional. As if the Angle *A* be equal to *a*, *B* to *b*; and moreover *A B*. *a b* :: *B C*. *b c* :: C D. *c d* than those two Polygons are Similar.



49. And among *Curvilineal* or *Mixt Figures*, those are *Similar* in which you may Inscribe, or about which you may Circumscribe similar Polygons; so that any Polygon being Inscrib'd or Circumscrib'd about one Figure, you may Inscribe or Circumscribe a Similar one about the other. For instance, if having Inscrib'd any Polygon as *A B C* D*E* about the greater *Curvilineal* Figure, you can Inscribe another in all respects Similar to it in the lesser Curvilineal Figure *a b c d e*, then those two Curvilineal Figures are Similar.

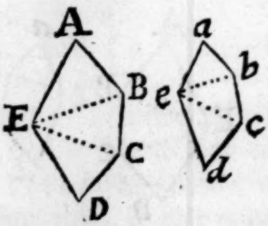


In like manner having taken two mixt Figures as the two Segments of Circles *B A C* and *b a c*; and having Inscrib'd in one any Triangle at Pleasure as *B A C*, if then you can Inscribe in the other Segment another Triangle *b a c* that shall be Similar to the former, then shall those two *Segments* be similar Figures. And if the Circles of which they are Segments be compleated, they shall be equal Parts of those two Circles, so that if *B AC* be a third Part of its Circle, *b a c* shall also be a third part of its Circle: And if to the Centres you draw the Lines *B* D and *C*D, and also *b d* and *c d*; the Angles D and *d* shall be equal. (See 4. 11. and the following Propositions.)



50. All Circles are similar Figures.

51. All similar Polygons may be divided into an equal Number of similar Triangles. Let the similar Polygons be *A B C D E* and *a b c d e*; and let the first be divided into Triangles by the Lines *E B* and *E C* (3. 14.) I say that if the other be also divided into Triangles by the Lines *e b* and *e c*, all the Triangles in one shall be (*respectively*) Similar to those in the other.

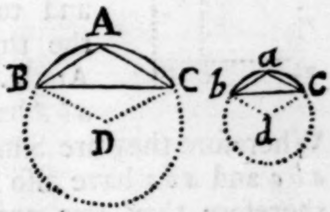


For

For Inſtance, I ſay the Triangle *abe* is Similar to *ABE*; for the Angle *a* is equal to *A* (by the ſuppoſition) and alſo *AB*. *ab*:: *AE*. *ae* (by the ſame) wherefore the Triangle *ABE* is Similar to *abe*. (6. 46.) Again, the Angle *EBC* may be proved equal to *ebc*; becauſe the Angle *ABC* is (by the ſuppoſition) equal to *abc* and it was proved (*in the laſt ſtep, where the Triangle* ABE *was proved Similar to* abe) that the Angle *abe* is equal to *ABE*; wherefore from equal things taking away equal, the Angle *EBC* remains equal to the Angle *ebc*. In like manner the Angle *ecb*, is prov'd equal to *ECB* and conſequently (6. 45.) the whole Triangle *ebc* will be Similar to *EBC*; and ſo of the reſt.

52. All ſimilar Polygons are to one another in a Duplicate *Ratio* of, or as the Squares of their Homologous Sides. I ſay as the Square of *AB* is to the Square of *ab*:: ſo is the whole Polygon *ABCDE* to the Polygon *abcde*. For ſince all the Triangles in one Polygon are Similar to thoſe in the other (6. 51.) All in one Polygon will be to all thoſe in the other in a Duplicate *Ratio* of any of their Homologous Sides; that is, as the Square of *AB* is to the Square of *ab*.

53. All ſimilar Figures, even Curvilineal ones, are to one another as the Squares of any ſide of any ſimilar Figures, which can be Inſcrib'd or Circumſcrib'd about them *v. gr.* Let there be two Circles, in which are Inſcrib'd two ſimilar Triangles *abc* and *ABC*: I ſay, the whole Circle *ABC*. is to the Circle *abc*:: as the Square of *BC*. to the Square of *bc*. or which is the ſame thing as the Square of the Radius *BD* to the Square of the Radius *bd*. For in or about the Circle *abc* may be Inſcrib'd or Circumſcrib'd any Polygon you pleaſe; (or at leaſt ſuch an one may be imagin'd) (4. 30.) But every Polygon inſcrib'd in *abc* will have a leſs *Ratio* to the Circle *ABC*, then the Square of *bc* hath to the



Square

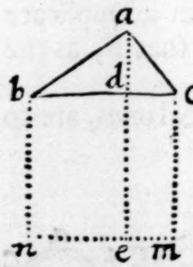
Square of *B C ι* and every one Circumſcrib'd about *a b c* will have a greater *Ratio* to the Circle *A B C*, as is eaſie to prove by the Precedent, and from what hath been ſaid of Circles in the fourth Book. Wherefore all ſimilar Figures, *&c.*

54. All this may be apply'd to Solids. And therefore *Similar Solids* are *ſuch*, as have their Angles all equal, and the ſides about thoſe Angles Proportional; or *(if they are of a Spherical or of any Spheroidical Figure) ſuch*, as can have ſimilar Solids Inſcrib'd or Circumſcrib'd in, or about them, *&c.*

55. Similar Solids are to one another in a *(Triplicate Ratio* of, or*)* as the *Cubes* (*of their Homologous Sides*, &c.*)* ſee 6. 36, 37, *&c.*

(*And therefore all Spheres muſt be to one another as the Cubes of their Diameters*, &c.)

56. If in a Rectangle Triangle *a b c* a Line as *a d* be drawn from the vertex or top of the Right Angle, Perpendicular to the Baſe, Hypothenuſe, or longeſt ſide *b c*: it ſhall divide the Triangle *a b c* into two other Rectangled ones, *a b d* and *d a c*, which will be ſimilar to each other and to the whole *b a c*. For 1. All the three Triangles have one Right Angle. 2. The Triangles *a b c* and *a b d* have the Angle *b* common to both: Wherefore they are Similar *(6. 45.)* 3. The Triangles *a b c* and *a d c* have alſo the Angle *c* common to both; therefore they two are Similar; *(and laſtly* a b d *and* a d c *being both Similar to one third Triangle* a b c, *will be ſo to each other.)*



57. The Perpendicular *a d* is a mean or middle Proportional between *b d* and *d c*. That is *c d. d a* : : *d a. d b.* For the Triangles *c d a* and *b d a* being Similar (by the laſt) *c d* (the leſſer Leg of the Triangle *c d a)* ſhall be to *a d* (the greater Leg) : : as the ſame *a d* (the leſſer Leg of the other Triangle *a d b)* is to *b d* the greater Leg. *(6. 46.)*

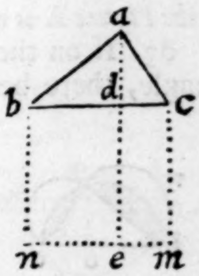
58. The

58. The Square of *a d* is equal to the Rectangle made between *c d* and *d b*. For, since *c d. d a* :: *d a. d b.* (by the last,) the Rectangle of the Extreams *c d* and *d b* is equal to the Rectangle of the mean Terms *d a* and *d a* (6. 28.) But the two sides of that Rectangle being equal; because 'tis only *d a* taken twice; that Rectangle must be the Square of *d a*; and so it may be laid down as an Universal Theorem, that, &c. (*The Square of the Perpendicular drawn from the Vertex of any Rectangle Triangle to the Hypothenuse, is equal to the Rectangle between the Segments of that Hypothenuse.*

59. The Square of a mean Proportional is always equal to the Rectangle of the Extreams.

60. To express a Rectangle you need use but three Letters. *v. gr.* When we say the Rectangle *b d c*: We mean a Rectangle one of whose sides is *b d*, and the other *d c*. But if we say the Rectangle *b c d*, we then mean a Rectangle one of whose sides is *b c* and the other *c d*.

61. In every Rectangle Triangle the Square of the Hypothenuse is equal to the (*Sum of the*) Squares of the two other sides (*or to the Sum of the Squares of the Legs.*)

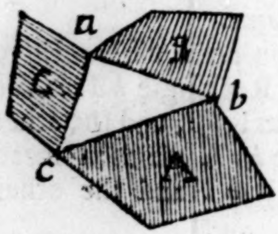


Let the Square *b m*, be divided by the Perpendicular *a d e* into the two Rectangles *d m* and *d n*. I say that the Rectangle *d m* is equal to the Square of *a c*; and the Rectangle *d n*, to the Square of *a b*: and that by consequence the whole Square *b m* is equal to the Sum of the Squares of *a b* and *a c*. For 1. The two Triangles *a d c* and *a b c* being Similar (6. 56.) *d c. c a* (in the lesser Triangle *d c a*) :: as the same *a c. c b* in the greater Triangle *a c b*. Wherefore *a c* is a mean Proportional between *d c* and *c b* (or *c m*) and consequently the Square of *a c* is equal to the Rectangle *b c d*, or *d c m*, that is *d m*.

And

And after the very same manner may *a b* be prov'd to be a mean Proportional between *b d* and *b c* (that is *b n*, &c.) (*for the Triangle* a b d *being Similar to* a b c; d b *the lesser Side in one will be to* b a *the greater side*; *as that* b a (*now the lesser side in the other Triangle* a b c) *is to* b c *the greater side: That is* d b. b a : : b a. b c. (*or* b n) *and consequently the Square of* a b *is equal to the Rectangle* d b n, *or* d n. *And so both the Squares together, of* b a *and* a c, *or their Sum, is equal to the Square of the Hypothenuse.* Q. E. D.

62. If upon the three sides of a Rectangle Triangle are made three similar Figures, and those similarily Posited; the greatest shall be equal to the other two.



For the three Figures being similar are as the Squares of their Homologous sides (6. 53.) And therefore the Figure *A* shall be to *B* and *C*, as the Square of *b c* is to the Squares of *a b* and *a c*. But the Square of *b c* is equal to those two Squares (*by the last*) therefore (*the Figure* A *is equal to both* B *and* C *together.*)

63. If on the Hypothenuse *b c* of a Rectangle Triangle, there be made a Semicircle *b a c* and on the other two sides *a b* and *a c* two more Semicircles *b n a* and *a m c* that great Semicircle will be equal to the other two, (by the last Proposition.) And if from the greater Semicircle, and the two lesser ones you take away what is common to both; which are the two shaded Segments *a b* and *a c*; what remains of each must be equal, *i. e.* the Triangle *a b c* is equal to both the *Lunes b n a* and *a m c*.

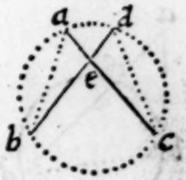
And this is the Quadrature of the *Lunes* of *Hippocrates* of *Scio*.

64. When the Triangle *b a c* is an *Isosceles* then the Lunes will be equal and then also the Triangle *a b o*, being the half of *a b c*, will be equal to each Lune. But if

if the Triangle be a Scalene as in this Figure, the Lunes are unequal; and 'tis as difficult to divide the Triangle *a b c* into two parts by the Line *a o*, ſo as to be able to prove the Triangle *a b o* to be equal to the Lune *b n a*, and the Triangle *o a c* to be equal to the other Lune *a m c*, this is I ſay, as difficult as to find the Quadrature of the Circle.

(N. B. *Since this, ſeveral ways have been diſcovered of Squaring any aſſign'd Portion of theſe Lunes. See the Philoſophical Tranſactions N.* 259. *pag.* 411.

65. Two Chords cutting or croſſing each other in a Circle have their Segments *Reciprocally Proportional*.



I ſay that *a e. b e* :: *d e. e c.* and conſequently the Rectangle *a e c* is equal to the Rectangle *d e b*.

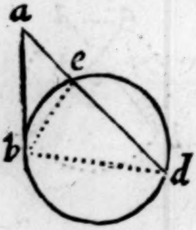
For draw the prickt Lines *a b* and *d c* and the two Triangles *a b e* and *d c e* will be Similar: Becauſe 1. The vertical or oppoſite Angles at *e* are equal (1. 23.) 2. The Angle *b* is equal to *c* becauſe ſtanding both on the ſame Ark *a d*, and being in the ſame Segment (4. 12.) wherefore the two Triangles are Similar, and conſequently *ae. e b* :: *d e. e c.* (6. 46.) Q. E. D.

66. If *a c* be the Diameter of a Circle, and *b d* a Perpendicular to it, *d e* or *b e* will be a mean Proportional between the Segments of the Diameter *a e* and *e c*. Becauſe *d e* is equal to *e b* (by 4. 6.) and therefore ſince by (*by the laſt*) Rectangle *b e d* (that is *b e* Square) is equal to *a e c*; as the Rectangles of the Parts of all croſſing Chords are, the Line *b e* or *c d* muſt be a mean Proportional between *a e*, and *e c*. Q. E. D.

67. Two Lines *a c* and *a d* drawn from a Point *a* without a Circle, to the internal and oppoſite Part of its Circumference; are to each other *Reciprocally* as their external Segments. I ſay *a c*, *a d* :: *a e. a b.* and

and consequently the Rectangle *c a b* is equal to *d a e*. For, supposing the Lines *c e* and *b d* to be drawn, the Triangles *a c e* and *a d b* will be Similiar, because the Angle *a* is common to both, and the Angle *c* is equal to *d* because standing on the same Ark *b e* (4. 12.) wherefore *d a*. *a b* : : *c a*. *a e* and Alternately *d a*. *c a* : : *a b*. *a e*. and by Inversion *c a*. *d a* : : *a e*. *a b* (6. 45.) And therefore the Rectangle *c a b* is equal to *d a e*. Q. E. D.

68. If one Line as *a b* touch a Circle, as in the Point *b*, and another Line *a d* drawn from the same Point *a* do cut it: Then is *a b* (*the Tangent*) a mean Proportional between *a d* and *a e* (i. e. *between the whole Secant and the Part of it without the Circle.*)



For, drawing the Lines *b e* and *b d*, the Triangles *a e b* and *b a d* will be Similar: Because the Angle *a* is common to both, and the Angle *a b e* (made by the *Tangent*, and *Secant e b*) is equal to *d* (*an Angle in the opposite Segment*) (4. 17.) therefore they are Similar; And consequently *e a* (in the little Triangle) will be to *a b* : : as that same *a b* is to *a d* in the greater Triangle: i. e. *e a*. *a b* : : *a b*. *a d*. Q. E. D.

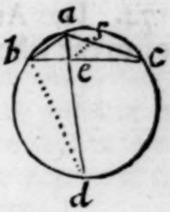
69. Let there be a Diameter *a b* cut in *c* by an Infinite Perpendicular *e e* whether within the Circle as in *Fig*. 1. at the Circumference as in *Fig*. 2. or without the Circle as in *Fig*. 3. Let there be drawn also from the Point *a* any Right Line as *a e* cutting the Perpendicular in *e* and the Circle in *d*. I say it shall always be as *a d*. *a c* : : *a b*. *a e*.

For drawing the Line *b d*, there will be made two Triangles that are Similar as *e a c* and *d a b*: which will be so, because they have one Angle as *a* common to both; and the Angle *d* equal

d equal to *c* because both are Right ones (for *d* is Right by 4. 14.) as being an Angle in a Semicircle and *c* is Right by the Suppoſition. Wherefore the Triangles are Similar and Conſequently *ad. ac :: ab. ae.* Q. E. D.

70. In the ſecond Figure *ab* is always a mean Proportional between *ae* and *ad*; and in the firſt, the middle Proportional is *aE* drawn from *a* to the place where the Line *ec* cuts the Circle.

71. If of a Triangle inſcrib'd in a Circle, the Angle *bac* be Biſſected by the Line *aed*.



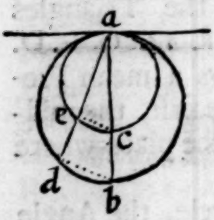
I ſay, then *ba. ae :: ad. ac.* For drawing the Line *bd*, there will be made two Triangles *abd* and *aec*; which are Similar; becauſe the Angle *d* is equal to *c* (4. 12.) as (*being in the ſame Segment*) or inſiſting on the ſame Ark, and *bad* is equal to *eac* by the *Suppoſition.* Wherefore the Triangles are Similar, and Conſequently *ba. ad :: ae. ac.* (and *therefore alternately* ba. ae :: ad. ac.) Q. E. D.

72. When the Angle at the Vertex is thus Biſſected, the Segments of the Baſe *bc*, are alſo Proportional to the Legs of the Triangle (*i. e.*) *be. ec :: ba. ac.* For ſuppoſing *ef* drawn Parallel to *ba.* Then will *ba. ac :: ef. fc* (6. 46.) But *ef* is equal to *af*: becauſe the Angle *aef* is equal to *eab*, (*as being Alternate Angles* 1. 31.) and conſequently to *eaf* (*by the Suppoſition*) wherefore the Triangle *aef* is an *Iſoſceles* (2. 15.) And therefore inſtead of putting it as before *ba. ac :: ef. fc.* we may ſay *ba. ae :: af. fc.* But as *af. fc :: ſo is be. ec* (6. 42.) wherefore *ba. ac :: be. ec.* Or which is all one *be. ec :: ba. ac.* Q. E. D.

(N. B. *This Propoſition is Univerſal; and if any Angle of a Triangle be Biſſected, the Legs about that Angle are Proportional to the Segments of the oppoſite ſide made by the Line Biſſecting the Angle.*)

73. If

73. If two Circles touch one another *(in a Point within)* as *a*, and if to that Point you draw a Tangent and a Perpendicular *acb* (which will paſs through both their Centres) (4. 5.) and if alſo you draw any Secant from the ſame Point as *aed*. I ſay, 'twill always be as *ae*. *ad* :: *ac*. *ab*. For having drawn the Lines *ec* and *db* the Triangles *aec* and *adb* will be Similar, as having the Angle at *a* common; and *e* and *d* both Right ones; (by 4. 14.) and conſequently *ae*. *ad* :: *ac*. *ab*. Q. E. D.



74. The Ark *ec* is to the Ark *db* as the whole Circle *aec* to the whole Circle *adb*. (6. 49, and 4. 11.)

BOOK

BOOK VII.

Of Incommensurables.

1. A Lesser Quantity is said to *Measure* a greater, when being taken a certain number of times, it is exactly equal to the greater. *v. gr.* Suppose a Fathom to contain six Feet; then may one Foot be said to *Measure* that Fathom, because being taken or repeated six times, it will be exactly equal to the Fathom.

2. The Quantity which is thus a *Measure* to a greater Quantity, is call'd a *Part* of that greater; and the greater Quantity is call'd the *Multiple* of the lesser. So, a Foot is the *Part* of a Fathom, and a Fathom is the *Multiple* of a Foot.

3. If you take the Quantity (*of a common* French *Pace)* which is two Foot and a half, and try with that to Measure a Fathom, you cannot do it: Because if you add that Pace only twice, it will make but five Foot, which are less than the Fathom; and if you take it three times, it makes seven Foot and an half, which are more than the Fathom; so that this Quantity of two Foot and an half cannot Measure the Fathom, and therefore properly speaking is not a *Part* of it. But nevertheless they may be said to be *Parts* of the Fathom, because this Quantity contains five half Feet; for an half Foot is a *Part* of a Fathom, because being taken 12 times it will just Measure it; so therefore this Pace contains *Parts* of the Fathom, because it contains five half Feet which are $\frac{5}{12}$ that is, five twelfths of a Fathom.

4. When

4. When two Quantities are ſuch, that a third can be found which ſhall be an (*Aliquot or Even*) Part of both, that is, which ſhall *Meaſure* them both exactly: Then thoſe Quantities are ſaid to be *Commenſurable*; As for inſtance, a Pace and a Fathom are two Commenſurable Quantities, becauſe we can find a third Quantity, *viz.* half a Foot, which will Meaſure them both; For if the half Foot be taken five times it makes the Pace, and taken 12 times, it makes the Fathom.

5. But when it is not poſſible to find any third Quantity which can Meaſure two others, then thoſe two Quantities are call'd *Incommenſurables.*

6. Commenſurable Quantities are as *Number to Number*; that is, thoſe Quantities can be expreſſed by Numbers; ſo that as one Quantity is to the other, ſo ſhall one Number be to the other. Thus a Line of ſix Feet or a Fathom, and a Line of two Foot and an half, as a Pace, are to one another as Number to Number. For half a Foot Meaſuring them both, the latter by being taken 5 times, and the former by being taken 12 times; its plain that one Line contains 5 half Feet, and the other 12, and therefore they are as 5 to 12, or as Number to Number.

7. If two Quantities are not as Number to Number, that is, if it be impoſſible to expreſs their Magnitudes by two Numbers, they are Incommenſurable: As is plain from the laſt.

8. We ought then to ſee whether there are in Reality any ſuch Quantities whoſe Magnitudes cannot be expreſs'd by Numbers, and if there be any *ſuch*, we muſt ſay that there are *Incommenſurable Quantities.*

9. *A Plane Number* is that which may be produced by the Multiplication of two Numbers (*one into another*). *v. gr.* 6 is a plane Number, becauſe it may be produced by the Multiplication of 3 by 2: For twice 3 makes 6: So alſo 15 is a plane Number ariſing from 5 being multiplied by 3; and 9 is a plane Number produced by the Multiplication of 3 by 3.

10. Thoſe

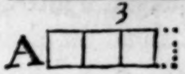
10. Those Numbers which being multiplied one by another, do produce a plane Number, are call'd the *Sides* of that Plane, as 2 and 3 are the Sides of the Plane 6; and 3 and 5 are the sides of 15.

11, If we imagine Unites to be little Squares, those Squares may be form'd into a Rectangle, if their Number be a Plane. *v. gr.* 12 Squares may be placed in the form of a Rectangle, one of whose sides may be 6 and the other 2. and 48 will make a Rectangle whose two sides may be 12 and 4. See the following Figures *B* and *C*.

12. *A Square Number* is a Plane, whose sides are equal; as 4, arising from the Multiplication of 2 by 2; as 9, the Product of 3 by 3; and 16 made by 4 multiplied by 4, &c.

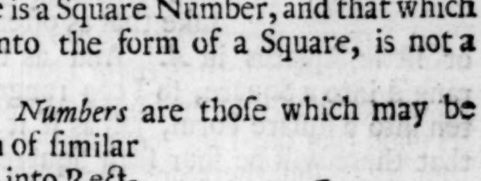
13. A Square Number may be rang'd into the form of a Square and that Number which can be ranged into the form of a Square is a Square Number, and that which cannot be rang'd into the form of a Square, is not a square Number.

14. *Similar plane Numbers* are those which may be ranged into the form of similar Rectangles; that is into Rectangles, whose Sides are Proportional; such are 12 and 48: for the sides of 12 are 6 and 2 (see *Fig. B*) and the sides of 48 are 12 and 4 (see *Fig. C*). But 6. 2 :: 12. 4. and therefore those Numbers are Similar.



15. All square Numbers are similar Planes (6. 32.)

16. Every Number may be placed in the form of a Right-line, and in that disposition may be taken for a Plane. Thus 3 (in *Fig. A*) may be conceived as a Plane Similar to 12 or *B*; For the



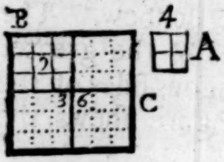
 sides

ſides of the Plane 3, are 1 and 3, (becauſe once 3 is 3) and the ſides of 12 are 2 and 6. But as 1.3::2.6.

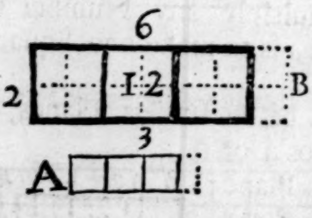
17. There are Numbers which are not *Similar Planes.* As if you examin from 1 to 10, you will find indeed that 1.4.9. being Squares are Similar, and ſo are 2 and 8, which have one ſide double to the other. But the reſt as 3.5.6.7. are by no means ſimilar Planes.

18. If one ſquare Number be multiplied by another, the Product will be a third ſquare Number.

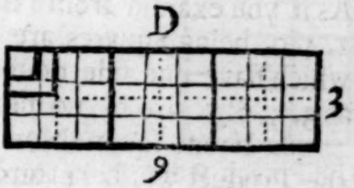
Thus *A*. 4. and *B*. 9, being both Squares do, when multiplied into one another, produce the Number 36 or *C*: And I ſay that third Number is a Square. For the meaning of multiplying *B* by *A*, is to take *B* as often as there are Unites in *A*. But I may conſider the whole Number *B*. 9. as one only Square, and I can take that as often as there are Unites, or little Squares in *A*. And as the Unites in *A*, are rang'd into a Square, ſo I can range the Square *B* as often into a ſquare Form, juſt as if it were an Unite. So that there will be four ſuch Squares of *B*, which being placed as you ſee in the Figure, will make the Square *C* or 36.



19. If two Numbers are ſimilar Planes, the greater may be divided into as many Squares, as there are Unites in the leſſer. *A*. 3. and *B*. 12. are ſimilar Planes; ſo that the ſide 3. is to 6:: as the ſide 1. is to 2. I can divide the Plane *B*. 12. into three Squares placed juſt in ſuch manner as thoſe little three Squares in the Plane *A*. And every one of the great Squares of *B* ſhall anſwer to 4 of thoſe in *A*. So alſo if the Planes had been 8 and 72; I can divide 72 into eight Squares, of which every one ſhall contain 9 of thoſe in the leſſer Plane 8. The ſame would come to paſs alſo, if either one, or both the Numbers had been



been Fractions. As if *A* contain 3 and $\frac{1}{2}$ and *B*. 14. I can divide 14 into three Squares and an half, dispoſed juſt like thoſe in *A*. as may be ſeen by the Partitions in the Figure and by the half Square added in Prickt Lines.



In like manner if the Planes were *B* 12 and *D* 27. I can divide 27, not only into three Squares, diſpos'd after the ſame manner as thoſe in *A*: But alſo into 12 Squares, ſo ranged as thoſe in *B*. as the prickt Lines in the Figure *D* do ſhew. The way to do which, is to divide the ſides of the greater Plane into as many Parts as the Homologous ſides of the leſſer Plane are divided into; the Figure ſhews the thing, and makes it eaſie.

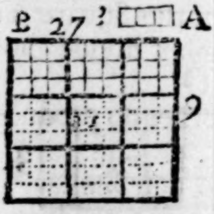
20. Thoſe plane Numbers which can be ſo divided as that there are as many Squares in the greater Plane, as there are Unites in the Leſſer, are Similar; this is the Converſe of the former.

21. Two ſimilar Plane Numbers multiplied one into another do produce a Square. For having divided the greater Plane into as many Squares as there are Unites in the leſſer (7. 19.) One Plane will be multiplied by the other, if the greater Squares of the greater Plane be taken as often as there are Unites or little Squares, in the leſſer Plane: But to multiply any Number of Squares, by the ſame Number, is to make one Square out of all thoſe Squares.

For Inſtance, *A* 3. and *B* 27. being ſimilar Planes, I conſider *B*. 27 as a Plane compos'd of three great Squares, as *A* 3. is a Plane compos'd, of three Unites, or three little Squares. So that if I take all theſe three great Squares, as often as there are Unites in *A*, that is three times; I produce then three times three ſuch great Squares as are in *B*,

 that

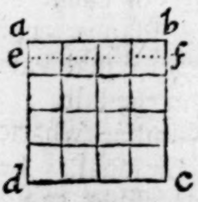
that is, 9 ſuch Squares; of which every one contains 9 of thoſe in *A*, and all theſe 9 Squares of *B* contain 81 of thoſe of *A*; ſo that *A*. 3. multiplying *B*. 27. produces 81. which is a Nnmber of the leſſer Squares rang'd into a Square Figure; and by Conſequence a ſquare Number (7. 13.) In like manner if the Planes were *B*. 12. and *D*. 27. I divide 27 into 12 Squares, which I multiply by 12, and there are Produced 144 greater Squares rang'd into the form of a Square, which do contain in all 324 of thoſe of the leſſer Plane. (N. B. *to divide* 27 *into* 12 *Squares, each Square muſt be* 2. 25. (*or two and a quarter*) *as you ſee it is in the Figure* D. N. 19.)



22. If two Plane Numbers are Similar, after what form ſo ever you range one, the other alſo may be ſo diſpos'd. Let 3 and 12 be ſimilar Planes. If 12 be ſo rang'd in a Right-line that it will make a Rectangle, one of whoſe ſides ſhall be 12 and the other 1. I ſay that 3 may be ſo diſpoſed as to make a ſimilar Rectangle, one of whoſe ſides will be 6 and the other the half of one, &c.

23. If one Number divide another that is a Square one, a third ſhall be produced which will be a Plane, Similar to the Diviſor.

Let there be a Square *ac* 16, and let it be divided by any Number, as ſuppoſe by 8. which is done if you take the eighth Part of the ſide *a d*, *viz*. *a e*, and thro' *e* draw the Parallel *e f*. For by that means you will have the Plane *a f*, which will be the eighth Part of the Square *a c*. But to divide a Number or a Plane by 8, is to take the eighth Part of that Number or Plane.



I ſay the Plane *a f*, is ſimilar to 8; for 8 being rang'd into a Right-line, ſo as to make a Rectangle, one of whoſe ſides ſhall be 8. and the other 1. ſhall be Similar to it, becauſe *a e* was taken the eighth Part of *a d* or *a b*: Where-

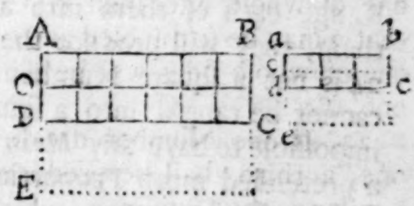
Wherefore as 8. 1 :: (which are the sides of the Plane 8 the Divisor) so shall *a b*. *ae* (which are the sides of the Plane of the Quotient arising when the Square *a c* was divided by 8.) Therefore if one Number divide another that is a Square, &c. Q. E. D.

24. If two Planes multiplying one another do produce a Square, those Planes are Similar.

25. Two Plane Numbers which are not Similar, if they are multiplied into one another, cannot produce a Square. These two Propositions are Consectaries from the foregoing ones.

26. If two Numbers are similar Planes, their *Equi-multiples* and any of their (*Respectively*) equal Parts, are allso similar Planes. Let the Planes be *a b c d*. 3. and *A B C D*. 12. so that *ab*. *A B* :: *b c*. *B C*. I say if you take the double of the one and the double of the other (or any other Equi-multiple, be it what you please) those Doubles shall be Similar.

For having taken *a e* double to *a d* and *A E* double to *A D*: in order to make the Plane *b e* double to *b d* and *B E* double to *B D*. 'Tis clear that *a d*. *A D* :: *ae*. *A E*. But *a d*. *AD* :: *a b*. *AB*. wherefore also *a e*. *A E* :: *ab*. *A B*. and consequently the Planes *b e* and *B E* are Similar.

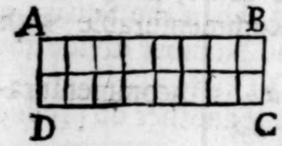


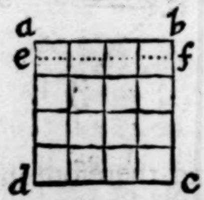
'Twould be the same thing had you taken their halves *b o* and *B O*, or any other equal Parts of each.

27. If two Numbers are not similar Planes, their Equi-multiples, and all their (*respectively*) equal Parts will also be not Similar, which follows from the last.

28. Between any two similar plane Numbers whatsoever, there is to be found a mean Proportional. Let the two Numbers be 2 and 8. I say 'tis possible to find a Number which shall be a mean Proportional between them. For if we imagine the Plane 8 to be ranged in a Right-line *AB*, and the Plane 2 also to be ranged in ano-

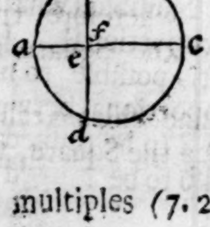
another Right-line as *AD*. and that out of those two Right-Lines there be formed the Plane *AC*. 16. That Plane *AC*. 16. will be produced by the Multiplication of the two Numbers 2 and 8 (6. 17. and the following Propositions) and consequently the Number of the little Squares of the whole Plane *AC*. 16. shall be a square Number (7. 21.) and they may be rang'd into the form of a Square (7. 13.) Let them then be dispos'd into the Square *ac*. So shall the Square *ac* be equal to the Plane *AC*. for 'tis only the same Number dispos'd or rang'd after another manner. Wherefore (6. 59.) the side *ab* 4 shall be a mean Proportional between *AD* 2. and *AB*. 8.





29. Between two Numbers Non-similar a mean Proportional can't be found. Let the Numbers be 4. and 6. Range each of them into a Right-line, and multiply them, they will produce the Plane 24. But this Plane 24 is not a square Number (7. 25.) and consequently cannot be rang'd into a square Form. Wherefore 'tis impossible to have any *Mean* between 4. and 6: For such a pretended mean Proportional must, multiplied by it self, produce a Square, which) as hath been prov'd elsewhere) will be equal to the Plane made between 4. and 6. (6. 59.) which is impossible, because this Plane 24, made out of 4 and 6. is not a square Number.

30. Let there be two Lines *ae* and *ec*, so to one another, as one Number to another Non-similar. *v. gr.* as 1. to 2. Let also *eb* be a mean Proportional, so that *ae*. *eb* :: *eb*. *ec*, I say that *eb* is *Incommensurable* with the two Extremes *ae* and *ec*. For *ae* and *ec* being as 1. to 2. (*i. e.*) as Numbers Non-similar (by the Supposition) as also are all their Equimultiples (7. 27.) 'tis impossible to find a mean Proportional



tional

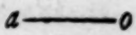
tional between *ae* and *ec* (by the Precedent and consequently, *eb* cannot be to *ae*, or to *ec*, as Number to Number. Wherefore it is Incommenſurable with them.

31. The Diameter of a Square *ab* is Incommenſurable to the ſide *ac*. For taking *ad* double to *ac* and making the Triangle *abd*; it ſhall be Similar to the Triangle *abc*; becauſe *cd* being equal to *cb* the Angle *d* is equal to *cbd* (2. 15.) and the Angle *d* muſt be a Semi-right one as well *cab*; wherefore *abd* is a Right-angle; and Conſequently *ac*. *ab*::*ab*. *ad*. That is, *ab*. is a mean Proportional between *ab* 1. and *ad* 2. and therefore Incommenſurable (by the Precedent.)

32. The *Power* of a Line is the Square, which is made upon it. Thus the Power of the Line *ac* (*Fig. preced.*) is the Square *aebc*; and the Power of the Line *ab* is the Square *abdf*. And we ſay that Line *ab* is double in Power (in Latin *bis poteſt*) to the Line *ac*. which is a manner of Speaking borrowed from the *Greeks*, and generally received amongſt *Geometers*.

33. The Diameter *ab* is *Commenſurable in Power* to the ſide *ac*: That is, 'its Square *abdf* is Commenſurable to the Square *aebc*; for 'tis indeed double to it.

34. But if you take *ao* a mean Proportional between *ab* and *ac*, that mean *ao* ſhall be Incommenſurable to them even in Power; *i.e.* the Square of *ao* is Incommenſurable to the Square of *ab* or to the Square of *ac*. for the Square of *ac* to the Square of *ao* is in a Duplicate *Ratio* of *ac* to *ao* (6. 29.) that is, as *ac* to *ab* (6. 30.) But *ac* is Incommenſurable to *ab* (7. 31.) wherefore the Square of *ac* is Incommenſurable to the Square of *ao*.



35. There is a *Second Power* of a Line which is call'd the Cube, which is made by multiplying the Square, by that firſt Line, or Root.

36. If

36. If two mean Proportionals *an* and *am* be taken between *ac* and *ab*; ſo that *ac*. *an* : : *am*. *ab*. the Line *an* will be Incommenſurable in this ſecond Power to *ac* (*i. e.*) The Cube of *ac* will be Incommenſurable to the Cube of *an*, becauſe the Cube of *ac* to the Cube of *an* is in a *Triplicate Ratio* of the ſide *ac* to the ſide *an*; *i. e.* as *ac* to *ab*. But *ac* and *ab* are Incommenſurable, wherefore, &c. However *ac* and *am* are Commenſurable in the ſecond Power, for the Cube of *am* is double to the Cube of *ac*.

a ——— *n*
a ——— *m*

37. 'Tis eaſie to apply to *Solid Numbers* what hath here been ſaid of Plane ones. And thoſe are call'd *Solid Numbers* which ariſe from the Multiplication of a Plane *Number* by any other whatſoever. *v. gr.* 18 is a ſolid Number made of 6 (which is a Plane) multiplied by 3; or of 9 multiplied by 2.

38. *Similar Solid* Numbers are thoſe, whoſe little Cubes may be ſo rang'd as to make ſimilar and rectangular Parallelopipeds.

39. *Cubick Numbers* are ſuch as can be rang'd into the form of Cubes as 8, or 27, whoſe *ſides* are 2 and 3 and their *Baſes* 4 and 9.

40. Every Cubick Number multiplying another Cubick Number produces a third Cubick Number.

41. Between two ſimilar ſolid Numbers there may be found two mean Proportionals.

That which hath been demonſtrated in reſpect to Plane Numbers may be applied to Solids.

42. Theſe Demonſtrations by which 'tis prov'd that there are Incommenſurable Lines and Magnitudes, ſhew alſo that a *Continuum* is not Compos'd of Finite Points: For if the Diameter as well as the ſide of a Square were compos'd of Finite Points, a Point would meaſure both the ſide and the Diameter, for that Point would be found a certain Number of times in the ſide and another determinate Number of Times in the Diameter, which the preceding Propoſitions prove impoſſible.

43. Be-

43. Becauſe in a Rectangle Triangle the Square of the Hypothenuſe is equal to the Sum of the Squares of the Legs; (*6. 61.*) we have always uſed this Triangle for the diſcovery of Incommenſurables. For if all the three ſides are Commenſurable they may be all three expreſs'd by three Numbers, and then the Square of the greateſt Number will be equal to the Sum of the Squares of the other two. As if the greateſt ſide or Hypothenuſe be 5 Feet, the leaſt ſide three, and the middle one 4: The Square of 5 will be 25, the Square of 3, 9; and the Square of 4 will be 16: And 9 and 16 added together do make the great Square 25. But if the leaſt ſide of ſuch a Triangle be 2. and the middle one 3. then the greateſt ſide cannot be expreſs'd in Numbers, becauſe the Square of the leaſt ſide 4 added to the Square of the middle ſide 9 make 13, which expreſs the Square of the greateſt ſide. But as that Number 13 is not a ſquare Number, ſo its ſide or Root cannot be expreſs'd by any Number.

44. At all times Men have been Sollicitous to find out ſome Method of diſcovering proper Numbers to expreſs the three ſides of a Rectangle Triangle, ſo as to be aſſured that all the three ſides are Commenſurable. Therefore I here ſhew you ſuch a Method, by which you may find out all the poſſible Numbers that are proper for this purpoſe.

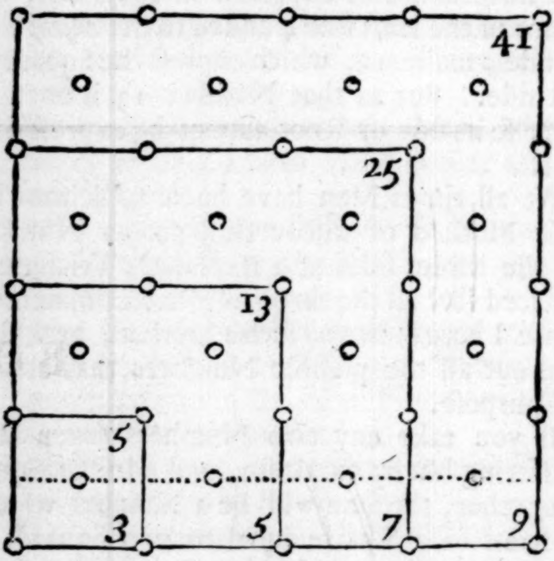
45. If you take any two Numbers (even Unity it ſelf) differing but by an Unite, and add the Squares of them together, the Sum will be a Number which ſhall be the Root of a Square equal to two Squares: And that Number will expreſs the greateſt ſide of a Rectangle Triangle, whoſe middle ſide ſhall be that Number leſſen'd by Unity, and the leaſt ſide ſhall be the Sum of the two firſt Numbers. *v.gr.* Having taken 1 and 2, and Squared each of them you have 1 and 4: Add thoſe two Squares together, and the Sum is 5. I ſay 5 will expreſs the greateſt ſide; and then 4 will be the middle one and 3 the leaſt: and 25 the ſquare of the Hypothenuſe will be equal to the Sum of the other two

Squares.

Squares. In like manner if you take 2 and 3, and add their Squares 4 and 9 together, the Sum is 13. Then I ſay will 13, 12 and 5 be the three ſides of a Rectangle Triangle; ſo that 169 the Square of 13 ſhall be equal to 144 and 25, the Squares of 12 and 5. Moreover if you take 3 and 4, the Sum of their Squares 9 and 16, makes 25; wherefore I ſay 25 may be the greateſt ſide of a Rectangle Triangle, whereof 24 will be the middle ſide and 7 the leaſt ſide.

All which may more eaſily be found after this manner,

46. If you range Unites into a ſquare Form as in the Figure; all thoſe Numbers which make a ſquare Figure



are proper Numbers to expreſs the greateſt ſide; the leaſt ſide ſhall be the Number contain'd in the two firſt Ranks of that ſquare Figure, and the middle ſide ſhall be a Number leſs than the greateſt, by one, or Unity.

47. If this Figure were continued it would give you all poſſible Numbers; But it muſt be obſerv'd alſo that the Equi-multiples of any 3 Numbers thus found will do the ſame thing: Thus, having found 5, 4 and 3 their doubles

10. 8. and 6. will repreſent the three ſides of a Rectangle Triangle, ſo that 100 the Square of 10 ſhall be equal to the Sum of 64 and 36 the two Squares of 8 and 6. And their Triples alſo 15. 12. and 9. will do the ſame thing: For any one may ſee that all theſe Numbers ſtill having the ſame Proportion, do as it were conſtitute but one only Triangle, *viz.* that which is expreſs'd by 5. 4. and 3. And therefore all thoſe Nnmbers may be taken for the ſame.

BOOK

BOOK VIII.

Of Progressions *and* Logarithms.

1. PRogression is a *Series* or Rank of Quantities which keep between one another any kind of similar Relation or Proportion; and every one of these Quantities is called a *Term*.

2. When the Terms which so follow one another do equally Increase or Decrease, the Progression is called *Arithmetical*; as are all Numbers proceeding according to the natural Order of the Figure, as 1, 2, 3, 4, 5, 6, &c. As also all odd Numbers, as 1. 3. 5. 7. 9. 11, &c. or as 4. 8. 12. 16, or as 20. 15. 50. and the like.

3. Arithmetical Progression may be encreased Infinitely, but not diminished.

4. If in an Arithmetical Progression there be four Terms, the Difference between the two first of which is equal to the difference between the other two those four Terms are said to be *Arithmetically Proportional*: As in the Progression of the natural Numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, &c. If you take four as 2. 3 :: : 9. 10 (*This mark* :: : *I shall for the future use to signifie Arithmetical Proportion*) there will be the same Arithmetical Proportion between 2 and 3, as there is between 9 and 10; that is, 10 exceeds 9, as much as 3 doth 2: so also 3. 5 :: : 8. 10. are in Arithmetical Proportion; and so are 1. 5 :: : 5. 9. where 5 being taken twice, is an Arithmetical mean Proportional between 1 and 9.

5. In

5. In Arithmetical Proportion the Aggregate or Sum of the two extreams is equal to the Aggregate of the two Means, as in 2. 3::: 9. 10. the Sum of 2 and 10 is equal to the Sum of 3 and 9, that is 12; so also in 3. 5::: 8. 10. The Aggregate of 3 and 10 is 13 which is equal to the Aggregate of 5 and 8. And the reason of this is self-evident: For tho' 10 exceed 8, yet that which is added to 8 (*viz.* 5.) doth just as much exceed 3 which is added to 10, and so there necessarily arises an Equality between them.

6. The Sum of the first and last Terms in any Arithmetical Proportion is equal to the Sum of the second, and the last save one; or to the Sum of the *third from the first Term, added to the third accounted backward from the last*, &c. as in the first Example, 1 and 9 make 10; and so do 2 and 8, 3 and 7, or 6 and 4 always make 10. And in the middle remains 5, which being taken twice (as if it were equivalent to two Terms because 'tis equally distant from the first and last Term) makes also 10.

7. If you add the first Term to the last, and multiply that Sum by half the Number of the Terms, the Product shall be equal to the Aggregate or Sum of all the Terms. As in the former Example, 1 added to 9 makes 10 and 10 multiplied by $4\frac{1}{2}$ (or 4. 5.) for there are 9 Terms, produces 45 which is the Sum of all the Terms from 1 to 9. As is manifest from the Precedent.

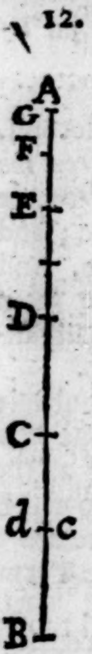
8. When the Terms of the Progression are continual Proportionals; that is, when the first is to the second :: as that is to the third Term : : as the third is to the fourth, and as the fourth is to the fifth, &c. then the Progression is call'd *Geometrical* as 1. 2. 4. 8. 16. 32 ∻. Or as 1. 3. 9. 27. 81 ∻ or again, as 3. 12. 48. 192. 768. or *descending*, as 8. 4. 2. 1 ∻ or lastly as $\frac{1}{2}$. $\frac{1}{4}$. $\frac{1}{16}$. $\frac{1}{32}$. $\frac{1}{64}$, &c.

9. Geometrical Progression may be encreas'd and diminish'd Infinitely.

10. When the Progression begins with 1, the second Term is call'd the *Root*, *Side* or first *Power*; the *third*, is call'd the *Square* or second *Power*; the *fourth*, the

the *Cube* or third *Power*; the *fifth*, the *Biquadrate* or fourth *Power*; the *sixth*, the *Sur-solid* or fifth *Power*; the *seventh* the *Quadrato-cube* or sixth *Power*, &c.

11. If (*in such a Progression*) you take 4 Terms, the former two of which are as much distant from each other, as the two latter are: Those are simply Proportional, and the Rectangle of their extreams is equal to that of their two middle Terms.



12. Let the Quantity *AB* be so divided in *C*, *D*, *E* and *F*, &c. that *AB*. *AC*::*AC*. *AD*::*AD*. *AE*, &c. Then I say *BC*. *CD*. *DE*. *EF*, &c. are in continual Geometrical Proportion; and also that *AB*. *AC*::*BC*. *CD*::*CD*. *DE*, &c. for because *AB*. *AC*::*AC*. *AD*. it will follow by Division of Proportion, that *AB* less *AC* (that is *CB*) *AC*:: as *AC* less *AD* (that is *DC*) *AD*. and consequently Alternately *CB*. *CD*:: *AC*. *A*D:: or as *AB*. *AC*. and so of all others it may be proved:: D*C*. *E*D:: *EF*:: *GF*, &c.

13. Let there be a Progression of Quantities in a Right-line *BC*, *C*D, D*E*, *EF*, &c. let *Cd* be equal to the *second* Term *C*D, that so we may have *Bd* the difference between the *first* and *second* Terms: And let it be made as *Bd*. *BC*::*BC*. to a fourth Line, *viz*. *BA*. I say that, if the Number of the Terms *BC*. *C*D. D*E*, &c. be Finite, tho' never so great, all those Terms taken together, although there be an hundred thousand Millions of them, shall be less than *BA*. But if we suppose the Progression infinite, or that the Terms are Infinitely many, then shall all of them taken together be exactly equal to *BA*. For since by the supposition *Bd*. (that is *BC* less *Cd* or *C*D) is to *BC*:: *BC*. (that is *AB* less *AC*) *AB*. it may easily be found that as *BC*. *C*D:: *AB*. *AC*:: *AC*. *A*D, &c. and consequently all the Terms *C*D, D*E*, *EF*, &c. will always be found within, or be hither the Point *A*. To which it Approaches the nearer, the more the Number of the Terms is increas'd. So that we see plainly, that all

all these Terms (which in Books are usually call'd *Parts Proportional*) tho' they be actually Infinite, cannot make an Infinite length, because they will all be included within the Line *B A*.

14. This Demonstration will appear much more easie and sensible by the Example of a particular Progression, where the Terms are in a double *Ratio*; *v. gr.* Let *C B* be double to D*C* and D*C* double to D*E*, &c. For if the Number of the Terms be here Finite, tho' it be an hundred thousand Millions, and you take the last and least Term, for Example *F E*, and add to it another Quantity, as suppose *A F*, equal to it: It is then plain that *E A* must be equal to the Term *E* D, which is the last save one; For *E* D is double to *E F* by the Supposition (the *Ratio* being every where double) and *E A* is also double to *E F* by the Construction, it having been made so, by taking *F A* equal to *F E*. In like manner *A E* with D*E*, that is *A*D, shall be equal to the following Term *C* D, and at last *AC* will be equal to *B C*. So that from hence it appears, that the first or greatest Term is always equal to all the others taken together, provided there be added to them but a Quantity equal to the last and least Term; but if nothing be added, the first Term is always greater than the Sum of them all.

If these Terms are suppos'd to be actually Infinite, then the greatest *BC* will be exactly equal to all those Infinite others taken together *C*D, D*E*, *E F*, &c. For any one may easily discern, that the more there are of such Terms, the more you approach towards *A*. by cutting off still the half of the Remainder: But when any Quantity is thus lessen'd by half, and the Remainder again by half, and then the half of that third Remainder taken and so on: 'Tis plain that by supposing the Diminution to be made an Infinite number of times, nothing at last will Remain.

This also might be demonstrated by a *Reduction ad Impossible*, by shewing that all those infinite Terms are, taken together, neither greater nor less than *B A*.

15. Hence may the Difficulties raised by the Schoolmen against the (*Infinite*) Divisibility of a *Continuum* be solved, tho' to Persons Ignorant of *Geometry* they appear Unsolvable: But indeed at the Bottom, they are nothing but meer Paralogisms.

16. If two Progressions are supposed, one Geometrical beginning with 1. and the other Arithmetical beginning with 0, so that the Terms in one shall be placed over and answer Respectively to those in the other. The Arithmetical ones are call'd *Logarithms Exponents* (or *Indexes*) as in the following Ranks

0. 1. 2. 3. 4. 5. 6. 7. 8.
1. 2. 4. 8. 16. 32. 64. 128. 256.

17. That which is produced in a Geometrical Progression by Multiplication and Division; is effected in the *Logarithms* by Addition and Substraction: As, if having three Numbers given 2. 8 : : 64; You would find a fourth Proportional to them in Geometrical Progression: You must multiply 64. by 8. (which are the two middle Terms) For the Product, 512. shall be equal to the Product made by 2. and the fourth Number sought, they being the two Extreams of four Proportionals. And to find this fourth Number, you need only divide 512 by 2, and the Quotient will be 256. So that 2. 8 : : 64. 256. and 64 and 256 will be just as far distant from one another in the Order of the Progression, as 2. and 8 are.

But if instead of the Geometrical Numbers 2. 8 : : 64. you had used their Logarithms 1 3 : : : 6 which Answer to them in the Progression, and were minded to find a fourth Logarithm, then you must have added 3 and 6 which make 9, and from thence have Substracted 1. there would remain 8. The Logarithm answering to the Geometrick Number 256.

18. So also, if there be two Geometrick Numbers 4 and 8, to which the Logarithms 2 and 3 do answer; by multiplying 4 by 8 you produce 32; the Number under the Logarithm 5, which is the Sum of the Logarithms of 2 and 3.

19. In like manner by multiplying 16 by it ſelf, there will be produced 256 which ſtands under the Logarithm of 8, the Sum of 4 added to it ſelf.

20. So if the Geometrical Number were required that ſhall anſwer to or ſtand under the Logarithm 16; you muſt take 256 which ſtands under 8, and multiply it by it ſelf, and it will produce 65536 the Number requir'd.

21. If moreover the Geometrical Number anſwering to the Logarithm 23 were required, you may take any two Logarithms whoſe Sum is 23, as ſuppoſe 7 and 16, and multiplying the Geometrical Number under them, *viz*. 128 and 65536 one by another, the Product will be 8388608. The Number which ought to ſtand under the Logarithm 23, or in the 23 place of a Series of Geometrical Proportionals beginning from 1.

22. From hence appears the way of Anſwering that ordinary Queſtion, how much a Horſe would coſt, if bought on this Condition: That for the firſt Nail in his Shoe a Farthing were to be paid, for the ſecond Nail two Farthings, for the third Nail four Farthings, for the fourth Nail eight Farthings, and ſo on, ſtill doubling for 24 times: For the 24th place in ſuch a Progreſſion, would be the laſt Number 8388608 Farthings, which being reduced is 8738 *l*. 2 *s*. 8 *d*. and being doubled according to (8. 14.) gives the whole price of the Horſe 17476 *l*. 5 *s*. 4 *d*.

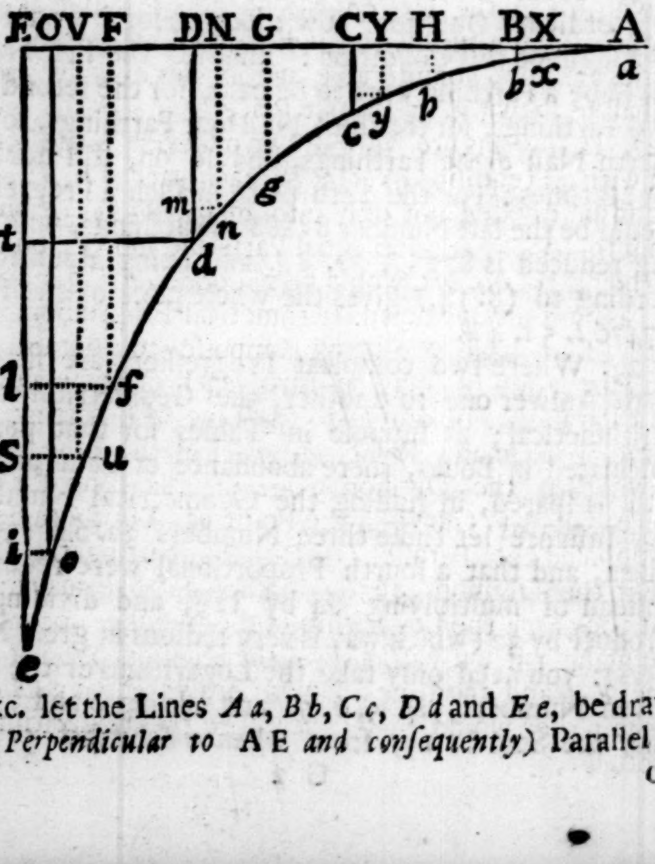
23. Where two compleat Progreſſions are fitted ſo as to Anſwer one to another, the Geometrical to the Arithmetical; as ſuppoſe in Tables for that purpoſe calculated in Books, there abundance of Pains and Labour is ſpared, in finding the Geometrical Numbers: For Inſtance let thoſe three Numbers 32. 64. 128. be given, and that a fourth Proportional were required: Inſtead of multiplying 64 by 128, and dividing the Product by 32 (which way is very tedious in great Numbers): you need only take the Logarithms of the three given Numbers, *viz*. 5, 6, 7. and adding 6 and 7 together the Sum in 13, from whence ſubſtract 5. there

 reſts

rests 8 which is the Logarithm of the fourth Geometrical Proportional sought, seek therefore in the Table for the Logarithm 8, and the corresponding Geometrical Number you will find to be 256.

24. But because in such a Geometrical Progression all Numbers will not be found, this Medium hath been discovered; they have calculated two Progressions, one of which, contains all Numbers 1. 2. 3. 4. 5. 6. 7. 8, &c. which seems to be an Arithmetical Progression, but yet hath in reality the Properties of a Geometrical one. And the other which contains Numbers in appearance the most Irregular, is neverthe less a true Arithmetical Progression. See here a Line, which will discover Perfectly all these Mysteries.

25. Let the Right-line *AE* be divided into the equal Parts *AB*, *BC*, *CD*, *DE*, &c. from the Points *A*, *B*, *C*, *D*,



E, &c. let the Lines *Aa*, *Bb*, *Cc*, *Dd* and *Ee*, be drawn all (*Perpendicular to* AE *and consequently*) Parallel to one

one another: And let them be all in a Geometrical Progreſſion: As, let *Aa* be 1, *Bb* 10, *Cc* 100, *Dd* 1000, *Ee* 10000, &c. Then ſhall we have two Progreſſions of Lines, the one Arithmetical and the other Geometrical: For the Lines *AB*, *AC*, *AD*, *AE*, are in Arithmetical Progreſſion, or as 1. 2. 3. 4. 5, &c. and ſo do repreſent the Logarithms; to which the Geometrical Lines *Aa*, *Bb*, *Cc*, &c. do correſpond.

26. Let each of the equal Parts *ED*, *DC*, *CB*, &c. be divided equally again in *F*. *G*. *H*, and let the Parallels *Ff*. *Gg*, &c. be drawn, and be mean Proportionals between the Collateral ones; that is, *Ee*. *Ff*::*Ff*. *Dd*. ::*Dd*. *Gg*, &c. Let there alſo be more mean Proportionals drawn from the middle of each Subdiviſion *EF*, *FD*, *DG*, &c. and ſo on, till theſe Parallel Lines growing very numerous, have at laſt but a very ſmall diſtance from each other; then imagine a Curve Line drawn thro' all the Extremities of theſe Parallels as *eou fdgba*: by this means you will gain a Line, whoſe properties are very conſiderable and its uſes equally great, as ſhall be ſhewn in its Proper Place.

27. If this Figure were drawn on a very large Table and with all requiſite Exactneſs; each part *AB*, *BC*, &c. might be divided not only into an 100 or 1000, but even into 10000, 100000 equal Parts and more. So that *AB* being 10000, *AC* would be 2000000, *AD* 3000000, &c. as muſt always be an Arithmetical Progreſſion.

28. The Line *Ee* being ſuppos'd to contain 10,000 Parts; let us imagine thro' each of thoſe Diviſions a Parallel to be drawn to the Line *AE* cutting the Curve in ſo many Points. *v.gr.* Let the Line *io* be drawn thro' the Diviſion 9900 of the Line *Ee* and which cuts the Curve in the Point *o*. Let there be alſo ſuppoſed the Parallel (to *Ee*) *Oo*, cutting the Line *AE* in the Diviſion 399563. Then any one may know that 399563 is the Logarithm of the Number 99000. In like manner if *Su* paſs'd thro' the Diviſion 9000 of the Line *Ee*, and the Line *uV* were drawn cutting *AE* in 395424 then would that Line *uV* be the Logarithm of 9000, &c.

 29. So

29. So that by this means a Table of Logarithms from 1 to 10,000 may easily be made; and farther by producing the Line *AE*.

30. Note, to obtain all the Logarithms from 1 to 10,000; 'twill be enough to seek the Logarithms from 1000 to 10000: That is (having drawn the Parallel *dt*) to take the Logarithms of all the Divisions from *t* to *e*, which Logarithms are all contained between *E* and *D*. For by this you will have the Logarithms of all the Parts that are between *t* and *E*; and whose Logarithms lye between *D* and *A*: For Example, since *Oo* is 9900 Parts, and its Logarithm 399563; the same Number may be taken for the Logarithm of 990 which is *Nn*; as also of the Number *Yy* 99, changing only the first Figure 3. Because according to the composition of this Line *ON* or *NY* ought to be equal to *ED* or *DC*, as any one may easily Prove. So that *ON* or *NY* will contain 100,000 and because *AO* is 399563, substracting *ON* 100,000, there will remain 299563 for *AN*. from whence also taking 100,000, there will rest 199563 for *AY*. And after the same manner, having *AY* 395424 for the Logarithm of *Vu* which is 9000; you may have also 095424 for the Logarithm of *Xx* which is 9. Or 195424 for the Logarithm of 90, or 295424 for the Logarithm of 900.

31. All this may be reduced to Practice for Calculation, without actually drawing these Figures, but only Imagining them to be drawn. For by the Rules of Common Arithmetick we may find out *Ff*, the mean Proportional between *Dd* and *Ee*, and after that, another Mean between *Dd* and *Ff*, or between *Ff* and *Ee*, &c. But what we have here explained is sufficient to gain as much Knowledge as is necessary for us to have of the Nature and Composition of Logarithms: There being no need for us to undergo the Labour of Calculating Tables of Logarithms, since it's already so well and so often done to our Hands. God, for the publick Good having raised some Persons, whom he was pleased to endow with sufficient Patience to surmount so tedious and laborious a Work, as one would think to be insuperable.

perable. For we know that above 20 Men were engag'd in such a Calculation for above 20 Years together with indefatigable Industry and Assiduity.

(Pardie *speaks here a little Covertly, seeming willing to Insinuate that this most useful and admirable Work was done first in his own Country; whereas the Logarithms were the Invention of my Lord* Neper *a* Scotch *Baron, and the* first Tables *were Calculated by him, with the Assistance of our Country-man Mr.* Henry Briggs.)

32. Besides these two kinds of Progressions, there is also a third which is call'd *Harmonical* or *Musical*: Where three Terms being taken, which immediately follow one another, it is found that the greatest is to the least : : as the difference between the greatest and the middle one, is to the difference between the middle one and the least, as 30. 20. 15. 12, &c. are in such an Harmonical or Musical Progression. For taking 30, 20 and 15. the difference between 30 and 20 is 10, and the difference between 20 and 15 is 5. But 10. 5 : : 30. 15. Therefore these are in Musical Progression.

33. This Progression may be diminished Infinitely, but cannot be encreas'd so.

" What is here said of this kind of Progression hath " no great use: And I dont design here to meddle with " uncommon and extraordinary things.

" But hereafter in the further prosecution of these Ele-" ments, I shall take notice of some properties of this " kind of Progression very well worth observing: Be-" cause they will give us some Light into what we " have of the Musick of the Ancients, about which we " are yet much in the Dark. There shall be Demon-" strated the Relation that the Hyperbola hath to this " kind of Progression. For as a Rectilineal Angle is " subservient to the finding out as many Arithmetical " Means as one will, between two given Quantities; and " as that Curve Line, which we have above described " for the Logarithms serves us also to discover as many " Geometrical Means as we please between two given " Quantities; so it shall then be shewed, that the Hy-

 " perbola

"perbola will help to find out between two given Quan-"tities as many Harmonical Means.

34. There is alſo a Progreſſion of Squares, Cubes, Biquadrates, Sur-ſolids, Quadrato-Cubes, &c. as 1. 4. 9. 16. 25. 36, &c. which are all Squares, whoſe Roots are the natural Numbers 1. 2. 3. 4. 5. 6, &c. ſo alſo 1. 8. 27. 64. 125. 216. are the Cubes of the ſame Numbers. Alſo 1. 16. 81. 256. 625. 1296, &c. are the Biquadrates of thoſe Numbers.

35. In a Progreſſion of Squares where 0. is the firſt Term as 0. 1. 4. 9. 16, &c. The Sum of all the Terms is greater than the third Part of the Product of the laſt Term multiplied by the Number of all the Terms; And this exceſs above the third Part, is always ſo much leſs, by how much the Number of the Terms is greater.

So in a Progreſſion of Cubes the Sum of all the Terms is greater than the fourth, as before: And in a Progreſſion of Biquadrate Numbers, 'tis greater than the fifth Part, and ſo on in the reſt. To prove which, it will be ſufficient to bring an Induction of particulars, as may be ſeen in this Table where the ſecond Row contains a Progreſſion of Squares from 0. and the third ſhews you

1	0	0	0	
2	1	1	2	$\frac{1}{2}$ or $\frac{1}{3} + \frac{1}{6}$
3	4	5	12	$\frac{5}{12}$ or $\frac{1}{3} + \frac{1}{12}$
4	9	14	36	$\frac{7}{18}$ or $\frac{1}{3} + \frac{1}{18}$
5	16	30	80	$\frac{9}{24}$ or $\frac{1}{3} + \frac{1}{24}$
6	25	55	150	$\frac{11}{30}$ or $\frac{1}{3} + \frac{1}{30}$
7	36	91	252	$\frac{13}{36}$ or $\frac{1}{3} + \frac{1}{36}$

the Sum of the Terms. *v. gr.* You may ſee here that the Sum of the Terms from 0. to 9. is 14. The fourth Rank contains the Product of each Term multiplied by the Number of the Terms from it to 0. which Number is put down in the firſt Rank; thus 36 is the Product of 9. multiplied by 4. The fifth Rank hath the Fractions exhibiting the Proportions of the Numbers in the third

third and fourth Rows: as right againſt 14 and 36 you have $\frac{7}{18}$ by which Fraction we would ſhew that 14 is to 36 :: as 7. is to 18. and that the Sum of the Terms, 14. is to the Product of 9. multiplied by 4. *viz*. 36: : as 7. is to 18.

Moreover in the ſame fifth Row after $\frac{7}{18}$ you ſee alſo theſe Characters (or $\frac{1}{3}+\frac{1}{18}$) by which we mean that $\frac{7}{18}$ is as much as $\frac{1}{3}$ added to $\frac{1}{18}$. Becauſe in reality $\frac{7}{18}$ are the ſame thing as $\frac{6}{18}+\frac{1}{18}$, that is as $\frac{1}{3}+\frac{1}{18}$. So that the Sum 14 is one third of the Product 36, and over and above contains $\frac{1}{18}$ of it.

So alſo it will be found that 30. which is the Sum of the Terms to 16 is more than the third of 80. which is the Product of 16 by 5. And that the Exceſs is $\frac{1}{24}$; For $\frac{30}{80}$ are $\frac{3}{8}$ or $\frac{9}{24}$ or $\frac{8}{24}+\frac{1}{24}$ or laſtly, $\frac{1}{3}+\frac{1}{24}$. But $\frac{1}{24}$ is not ſo great as $\frac{1}{18}$: So that we ſee in the Continuation of this Table, that theſe Exceſſes which are above the third Part of the aforeſaid Product do continually decreaſe, as the Number of the Terms increaſes: For theſe Exceſſes will be $\frac{1}{18}\,\frac{1}{24}\,\frac{1}{30}\,\frac{1}{36}\,\frac{1}{42}\,\frac{1}{48}$, &c. The Denominator of the Fraction always encreaſing by the Number ſix.

36. If ſuch a Table were made for Cubes, it would be found that the Fractions which are the Exceſſes above the fourth Part (as before) would alſo continually decreaſe in value, their Denominator encreaſing by 4 as often as any new Term ſhould be added to the Progreſſion. And thus it will be found to be in all other Progreſſions (*of theſe kinds*) if ſuch like Tables be made; as was ſaid in General, in the Precedent Propoſition.

All which are of great uſe in the ſubſequent Parts of *Geometry*: Where we ſhall treat alſo of many other kinds of Progreſſions.

BOOK

BOOK IX.

Problems *or* Practical Geometry.

1. THat Propofition is call'd a *Problem* in (Geometry) which teaches us how to *do* any thing, and demonftrates alfo the Practice of it: Whereas *Theorems* are Speculative Propofitions, in which are confider'd the Affections and Properties of things *already done*.

2. *From a given Point* a *in the Line* c a b, *to erect a Perpendicular.* Take, each way, from the Point *a* the two equal Parts *a c* and *a b* (it matters not whether they be great or fmall provided they be but equal). Then open the Compaffes a little more, and on the Points *b* and *c*, as on Centres, ftrike over the Point *a*, two little fimilar Arks, croffing each other in the Point *d*. Then lay a Ruler to the Points *a* and *d*, and draw the Line *d a*, it will be the Perpendicular requir'd (2. 16).

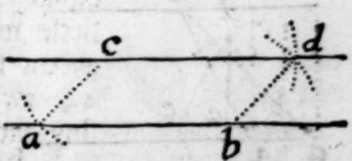
3. *From a Point given* d *to let fall a Perpendicular to the Line* b a c. On *d* as a Centre ftrike the Ark of a Circle cutting the given Line in the two Points *b* and *c*: Then with the fame opening of the Compaffes ftrike, below the Line, two fmall obfcure Arks croffing each other in the Point *e*. Draw the Line *d e*, which fhall be the Perpendicular required. (2. 16.)

4. When

4. When the given Points *a* or *d* are near the end, or ſide of your Paper, or the Surface on which you work, where the Figure ought to be drawn: ſo that you cannot come to take a reaſonable diſtance beyond the Point *a*; as you could in the preceding Caſe, where it was near the middle of the Line; but we now ſuppoſe it be to near to the end. Then you muſt proceed thus. If the Point *a* be given in the Line, take any Point as you pleaſe towards *e*, and from thence as on a Centre, draw a Circle which ſhall paſs thro' *a* and alſo cut the given Line in *b*. Then from *b* thro' *e*, draw the Diameter *b e d* cutting the Circle in *d*. The Line *d a* ſhall be a Perpendicular to *b a* (4. 14.)

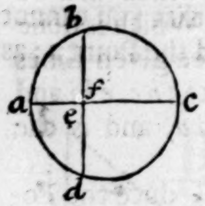
But if the Point *d* were given without the Line (*any where either above or below it*) draw a Line at Pleaſure as *d b* and from the middle of that Line (*e*) draw a Circle which ſhall cut the Line *b a* in *a*: Then will *d a* be the Perpendicular required. (4. 14.)

5. *From a Point given to draw a Parallel to a Line given.* Let the Line given be *a b*, and the Point *c*, thro' which the Parallel is to be drawn. On the Point *c* as on a Centre ſtrike the Ark of a Circle cutting the Line given in the Point *a*: Then ſet the Foot of the Compaſſes down any where (at a good diſtance from *a*) in the given Line, as at *b*, and keeping them at the ſame diſtance as before ſtrike the Ark *d*: Then take in the Compaſſes the Length *a b*, and putting one foot in *c* draw an Ark cutting the other in the Point *d*. Lay a Ruler thro' *c* and *d*, and you will draw the Line *c d* Parallel to *a b*. For the Quadrilateral Figure *a c d b* hath its oppoſite ſides, equal by the conſtruction, and conſequently is a Parallelogram, by the converſe of the 9*th* Prop. of the third Book.



6. *Between*

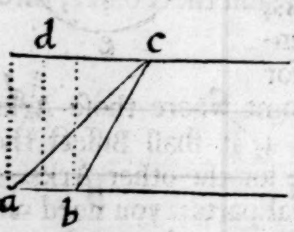
6. *Between two Lines given* a e *and* e c *to find a mean Proportional.* Place the two Lines given so that they may make but one Right-line as *a e c* which Bisect in *f*, and on that Point *f* describe the Circle *a b c*. Then erect the Perpendicular *e b* cutting the Circumference in the Point *b*; which shall be the mean required; for *a e. e b : : e b. e c.* (6. 66.)



7. *To make a Square equal to a Rectangle given.* Find a mean Proportional between the sides of the Rectangle and the Square of that shall be equal to the Rectangle.

8. *To three Right-lines given, to find a fourth Proportional.* Let the three Lines given be *a d*, *d e* and *a b*, set *a d* on *a b* and let *d e* be set transversely so as to make an Angle with *a d*, and join *a e* so as to make the Triangle *a e d*; produce *a e* to *c*, and thro' *b*, draw *b c* Parallel to *e d*. I say *b c* is the fourth Proportional sought: For *a d. d e : : a b. b c.* (6. 43.)

9. *To make a Rectangled Parallelogram* a d, *equal to a Triangle given* a c b. Thro' the Top or Vertex of the Triangle *c* draw a Line Parallel to the Base *a b* and make a Rectangle on half the Base *a b*, having the same height with the Triangle given, that Rectangle shall be equal to the Triangle: Because one on the whole Base *a b* and between the same Parallels with the Triangle, would be double to it by 3. 18.



10. *A Rectangle being given to make another Rectangle equal to it which shall have a length given.* Let the Rectangle given be *a c* and let it be required to make another equal to it, the length of one of whose sides shall be the Line *e f*: here are now 3 Lines given, *viz.* *a b* and *b c* (which are the sides of the Rectangle given) and *e f*, which

which muſt be one ſide of the Rectangle required. Therefore a fourth Line muſt be ſought, which ſhall be the other ſide of the Rectangle ſought: Which is done by finding a fourth Proportional to the three given Lines (9. 8.) which let be *e h*. So that *e f. a b* :: : *b c. e h*: and then I ſay the Rectangle *f h* is equal to *d b*, and is the Rectangle requir'd. (6. 27.)

11. *To Square any Polygon whatever.* Reduce the Polygon into Triangles (3. 22.) and make Rectangles ſeverally equal to each of thoſe Triangles (9. 9.) and order it ſo that thoſe Rectangles may have all the ſame length (9. 10.) Then join all thoſe Rectangles together ſo as to make one great one, equal to them all; and laſtly make a Square equal to that Rectangle (9. 7.) and 'tis done.

12. *To divide a Circle into four and into 6, and all Arks into two equal Parts.* To divide it into four Parts draw two Lines as *d a c* and *B a e* at Right-angles to each other. To divide it into eight Parts; Biſſect the four Arks *B c. c e*, &c. which is done by ſtriking (without the Ark *B c*) two other Arks, with the ſame opening of the compaſſes, from the Points *B* and *C*, for

if a Line be drawn, from the Point where thoſe Arks croſs each other, to the Centre *a*, it ſhall Biſſect the Ark *B C*. The like is to be done for the other Arks.

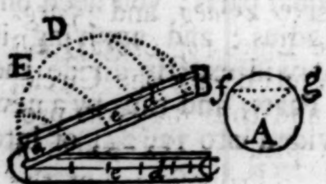
To divide a Circle into ſix equal parts; you need only take the length of the Radius: and applying it ſix times about the Circle, it will exactly divide the Circumference into ſix equal Parts, and thus by a new Biſſection, may a Circle be divided into 12, 24, or into 48 equal Parts, &c.

13. *To divide a Circle into five, into fifteen, and into other Equal Parts.* This may be done thus; (as I demonſtrate in *Algebra.*) Make a Rectangle Triangle, one of whoſe Legs ſhall be the Radius of the Circle and the other half the Radius. From the Hypothenuſe of

this

this Triangle, take half the Radius; the remainder ſhall be the Chord of 36 *deg.* and the ſide of a Decagon. Double that Ark, you have the Ark of 72 *deg.* (*whoſe Chord is the ſide of a Pentagone*) and it is the fifth part of the Circumference; and the ſaid Chord ſhall be alſo the Hypothenuſe of a Rectangle Triangle, one of whoſe ſides is the Radius and the other the ſide of a Decagon. And as by the laſt was found the Chord of 60 *deg.* ſo by Subſtracting the Chord of 36 *deg.* from 60 *deg.* you may have the Chord of 24 *deg.* which is the 15 part of the Circumference. But for Practice, the ſhorteſt and ſureſt way, is by repeated Trials with the Compaſſes to find a Diſtance that will go preciſely five times about the Circle: Then divide, after the ſame manner (by Tryals) that diſtance into three equal parts exactly: So ſhall you gain a Chord that will divide the Circumference into 15 equal Parts, and then dividing each of thoſe 15 Chords into four equals Parts, and each of thoſe into ſix; you will divide the whole Circumference into 360 *deg.* And this Diviſion is moſt Commodious for Practice and Uſe. Note, that the way to divide a Circle into 3, 5, 7, or into any other odd Number of Parts is not yet found Geometrically; Geometrically I ſay, that is, by making Uſe only of a ſtrait Line and a Circle.

" This Diviſion of a Circle into 360 *deg.* is very uſeful, when a Perſon underſtands how to uſe the *Compaſſes of Proportion* (or *Sector*). 'Tis ſo called becauſe 'tis a kind of Compaſſes with broad Legs: as *a B. a C* on which are deſcribed divers Lines and Diviſions; But thoſe which are moſt in uſe, are of two ſorts.



" On one ſide of this Sector, and on each Leg, is a Line *a e B* and *a e C*, which ſerves to divide a Circle into 360 *deg.* at once, and alſo to take at any time as many Degrees as you pleaſe. And this Line on the Sector is thus divided.

14. *To*

14. *To divide and graduate the Sector, that it may serve for the Division of a Circle.* Imagine a Semicircle *a E D B* accurately divided into 180 *deg.* if then from the Point *a*, as from a Centre, you transfer the Divisions of the Semicircle into the Line *a B.* *v. gr.* If from *E*, 60 *deg.* you draw the Ark *E e*, and if from *D*, 90 *deg.* in the Semicircle, you draw the Ark *D d*, &c. Then ought 60 *deg.* on that Leg of the Sector, to be placed at the Point *e*, and 90 *deg.* at the Point *d*, *&c.* And if you Transfer the same Degrees after the same manner into the other Leg *a C*, you will graduate the Lines *a B* and *a C* (on the Sector) as they ought to be for this purpose; *and they will be two Similar Lines of Chords.*

15. *To explain the Use of the Sector as far as it serves for the Division of a Circle.* Let there be a Circle given *Af.* take with your Compasses the Radius *Af* and (*keeping that distance*) set one Foot of them in *e* or 60 *deg.* on one Leg of the Sector; move the other Leg of the Sector to and fro so long, till the other Point of the Compasses falls exactly on *e* or 60 *deg.* in that Leg of the Sector: So that the Distance *e e* be exactly equal to the Radius *Af.* Then if you would have readily 90 *deg.* of that Circle; (*Letting the Sector lye still, and always keeping the same Angle*). Open your Compasses till the Points fall exactly on *d* and *d* or 90 *deg.* and 90 *deg.* on each Leg of the Sector: And then that distance transferr'd into the Circle, in *fg*, gives you the Ark of 90 *deg. fg.* So also if you would have had 35 *deg.* you need only apply your Compasses to 35 *deg.* and 35 *deg.* on each Leg of the Sector in the Lines (*of Chords*) *a B* and *a C*; and that distance transferr'd into the Circle, shall cut off the Ark of 35 *deg.* and thus may you proceed to find any Degrees you please. All which is grounded on the 42, 43, 49 and 50 Propositions of the VI Book. For since all Circles are similar Figures, (6. 50.) the Chord *fg* will be to the Radius *f A* : : as the Chord *d d* to the Radius *e e*; that is, as *a d* is to *d e.* Now 'tis plain from what hath been prov'd elsewhere, that the Triangles *a d d* and *a e e* are Similar; and there-

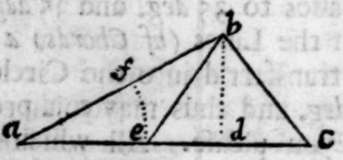
fore

fore *dd. ee* :: *ad. ae.* But *dd* is by the Construction equal to *fg.* and *ee* to *Af.* wherefore *fg. Af* :: *ad. ae.* Q. E. D.

16. *To divide the Line of* Equal Parts *or* Lines *on the Sector, for the Division of any Right-lines given.* There being two Right-lines drawn from the Centre of the Sector on the Legs as *aB* and *aC*: Let each be divided into 100 or 200 equal Parts: And then they will serve

to divide any Line given into any Number of equal Parts; As for Instance, let the Line given be *cb*, and that you were required to take $\frac{25}{97}$ parts of it. Now to divide the whole Line *cb* into 97 equal Parts and then to take 25 of them according to the Common way of dividing Lines, would be very tedious: But by the Sector 'tis done easily and speedily thus, take the length of the whole Line *cb* in your Compasses and fit, or apply it over in your Sector between 97 and 97 in each Leg, from *B*, suppose to *C*. Then letting the Sector lie open'd at that Angle, take in your Compasses the distance between 25 and 25 in each Leg or between *e* and *e* which transfer into the given Line from *b* to *f*; so shall *bf* be just $\frac{25}{97}$ of the whole Line *cb*: As is plain from the Triangles *ABC* and *Aee* being Similar.

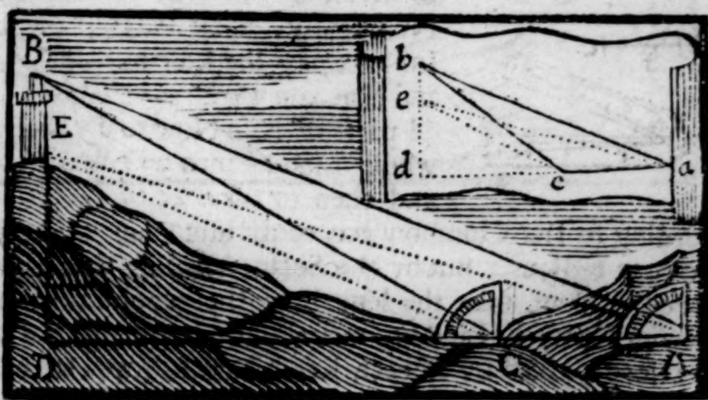
17. *On a Line given to make an Angle that shall contain any Number of Degrees assign'd.*



Let the Line given be *ac*, on which 'tis required to make an Angle of 30 *deg.* From the Point *a* as from a Centre strike the Ark *fe*, from which take by the Sector, or otherwise, 30 *deg.* from *e* to *f*; then thro' *f* draw the Line *af* which with the Line *ac* will make an Angle of 30 *deg.*

18. *Having*

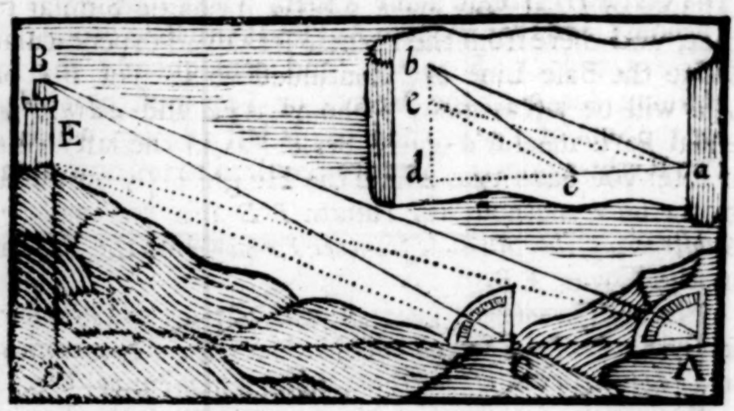
18. *Having the Angle of any Triangle and one ſide given to find the other two ſides.* Suppoſe you are told there is a certain Triangle ſome where, whoſe Baſe *AC* is 10 Fathom; and that the two Angles at the Baſe are *ACB.* 150 *deg.* and *CAB* 20 *deg.* (and conſequently the remaining Angle at the Vertex or Top muſt be 10 *deg.* for the Sum of 150, 20 and 10, is juſt 180 *deg.* which is two Right-angles.) You are required to tell how many



Fathom there are in the other ſides *AB* and *AC.* Make on Paper, or rather on fine Paſtboard, a Triangle *abc* Similar to the propos'd one after this manner. Take a Baſe at pleaſure *ac* and from any Scale of equal Parts let it be 10 Inches, half Inches, &c. in Length. On this Line *ac* make two Angles, one *cab* of 20 *deg.* and the other *acb* of 150 (9. 17.) Then will the two Lines *ac* and *cb* croſs one another when produced in the Point *b.* Then meaſure *(on the ſame Scale you took the Baſe* a c *from)* how many *Inches*, &c. the Lines *ab* and *cb* are in Length: And you may be aſſured that there are juſt ſo many Fathom in the Lines *AB* and *CB* ſought, as you find Inches, &c. or any equal Parts, in the Lines *ab* and *cb.* For ſince the Triangles are Equiangular, they are Similar and therefore *ac.ab* :: *AC.AB*, &c.

19. *To Meaſure Diſtances, Heights, Depths, and in General, the Magnitudes and Dimenſions of all Remote and*

Inacceſſible Places. If on the top of any Hill appearing at a diſtance there were a Tower as *B E* and its diſtance



from us and its Height were required: You muſt firſt with ſome Inſtrument (as with a *Quadrant*, that is the fourth Part of a Circle divided into 90 *deg.* and furniſh'd with a Ruler, or Label with Sights, and moveable on the Centre) you muſt I ſay with ſome ſuch Inſtrument, take two Angles at two ſeveral Stations in this manner: If you are in the Station *A*, place your Inſtrument ſo, that one ſide of it may anſwer exactly to the Horizontal Line *A D*; and keep it without raiſing or depreſſing it in this Poſition. Then place your Eye at *A*, (that is, at the Centre of the Inſtrument,) and turn the Label till it Point to the Top of the Tower *B*, and that looking thro' the Sights you can ſee the top of the Tower exactly; then will the Label cut in the Limb of the Quadrant the Degrees of the Angle *B A D*, for the Limb is ſuppos'd to be graduated for this purpoſe. Then change your Station, moving in a Right-line forwards 10 Fathom (or it might have been any other Diſtance, and backward as well as forward) to *C*. and there take after the ſame manner the Angle *B C D*: By which means you will have alſo the Angle *B C A*, becauſe thoſe two together make 180 *deg.* or two Right ones. So that in the Triangle *A C B* you have now found the Baſe *A C* which is 10 Fathom, and alſo the two Angles at the Baſe;

and

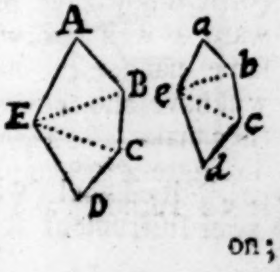
and consequently the sides *C B* and *A B* may be known (9. 18.) And then you may have the Height *D B* or the distance *A D*, if you make a little Triangle Similar to that, and there from the Point *b* let fall a Perpendicular *b d* to the Base Line *A C* continued to *d*. For *B D* or *A D* will be just as many Fathoms as *b d* and *a d* will be equal Parts measur'd on the Scale. (as in the last) And if after you have thus gain'd the Height *B D*, you find, by the same method, the Height *E D* also, you may (*by Substracting this Altitude from the Former*) find the Height of the Tower *E B*.

N. B. (*The common Quadrant with a String and Plummet and with the Sights fixt on one of its sides, is more convenient and ready than this of* Pardies, *which is now out of Use.*)

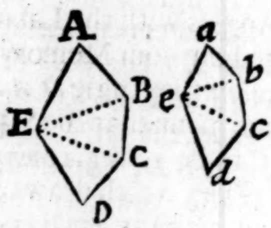
" Some times instead of advancing towards the Tower " and of making Observations of the Height below; " or of those Angles the visual Rays make with the Ho- " rizontal Line, it is convenient to take two Stations " sideways of each other; But it comes all to the same " and the Practices in reality are not at all different.

" And by this means, as any one may see, may all " imaginable Heights and Distances and other Dimen- " sions be taken; provided we can but come to observe " their Extremities, from two different Places. I shall " not stay now to describe the Particular ways of doing " this; nor to enumerate the great Advantages that " would accrue from the use of Teloscopical Sights " fixt on the Label, or on the side of the Instrument " used in taking Angles which indeed is an Invention of " inestimable Benefit to Surveyors.

20. *To take the Plane of any Place.* Let *A B C D E* be a City, or any other Place, and you were required to take the Plane, and to make a Draught of it. Take all the Lengths of its sides, and of Lines drawn from Angle to Angle: And transfer all these upon Paper, laying them down according to their true Proporti-

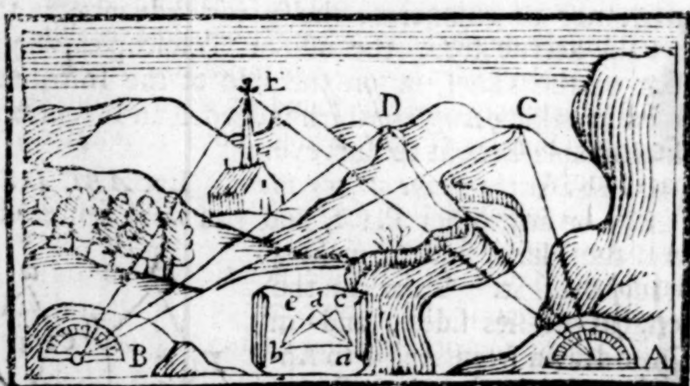


on; For Inftance, having found AB to be 30 Paces BC to be 59 CD to be 50: BE to be 67, and AE 49, &c. and having ready drawn on Paper a plain Scale divided into 100 equal Parts. Make the Line ab 30 of fuch Parts; be, 67; and ae, 49; then thofe Lines drawn and join'd together will make the Triangle abe every way Similar to the Triangle ABE. And if you go on thus and make the Triangle bec Similar to BEC, &c. you will form the Figure $abcde$ every way Similar to the Plane of the Place $ABCDE$.



21. But if you cannot get into the Place to Survey it and to Meafure the diftance between the Angles EB and EC, you muft take the feveral Angles of the Plane, and transfer them into your Draught; fo that if the Angle BAE be 66 *deg.* the Angle bae muft alfo be 66 *deg.* and fo of all the reft.

22. *To make a draught of any City or Country.* Afcend up into any two elevated Places from whence you can plainly fee the City or Country whofe Delineation you would make. And having with you a Quadrant, whole Circle, or Semicircle well divided into Degrees, together



with its Label (with Sights) upon its Centre: Place your Inftrument at A, and fo that one of its fides may

lye

lye in a Line between *A* and *B*, which done, and the Instrument fixt there, obſerve the ſeveral Steeples, eminent Houſes, Towers, Hills and all other Remarkable Places as *E D C*, *&c.* and take their Angles with the Label and Sights, and write them all down to help your Memory. Thus, let the Angle *C A B* be 50 *deg.* 30 *min.* the Angle *D A B* 45 *deg.* 8 *min.* &c. Proceed after the ſame manner at the Station *B*; noting down the Angle *A B C* to be 40 *deg.* 10 *min.* the Angle *A B D* 47 *deg.* 28 *min.* &c. After which draw on Paper any Line at Pleaſure as *a b* and make at each end of it Angles equal to thoſe which you found *c a b* equal *C A B*, *d a b* equal to *D A B* and *a b c* equal to *A B C*, &c. and by this means you will have the Points *c*, *d* and *e*, &c. which will be in the ſame Poſition to one another as the Steeples, or other eminent Places *C D E*, &c. are. And thus having drawn the moſt Conſpicuous and Principal Places the reſt may eaſily be taken by the Eye. But to make this Operation very Exact, 'tis convenient to take the Angles alſo at a third or fourth Station, and then if they all agree, any one will know that the Work was well done.

23. *Having the two ſides of a Triangle and the Angle between them given, to find the third Side and the two other Angles ......*

24. *Having two ſides, and an Angle oppoſite to one of them given to find the other Side and the remaining Angle ......*

25. *All the Angles and one ſide given to find the reſt ......*

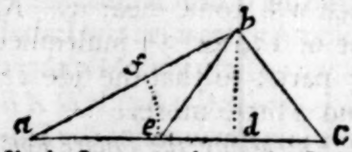
26. *The three ſides being given to find the Angles ......*

All which may eaſily be found by drawing on Paper a Triangle Similar to that whoſe Sides or Angles are in part ſought.

27. To meaſure the *Area*, (that is) the Capacity of, or ſpace contain'd in any Right-line Triangle given *a b c*. From the Top *b* let fall a Perpendicular *b d* to the Baſe *a c*, (produced if there be occaſion.) Then divide *a c* into 10, (or any other equal Parts you pleaſe) and find how many ſuch parts are contained in *b d*, for then multiplying the half of *b d* by 10, you will have the Area of the Triangle (3. 18.) as if *d b* contains four

H 3 ſuch

such parts of which *a c* contains 10. You multiply 10 by 2 (half of 4) and the Product will be 20: which is the Area of the Triangle *a b c*. That is, the Triangle contains as much space, as 20



little Squares do, whose Root or Side is the 10th part of the Line *a c*.

With Regard to Practice, there is no method easier nor exacter than this: (Only there is no need of actually Dividing *a c* into 10 or any other number of Parts as above: For having a good Diagonal Scale, you need only measure there the length of the Base and Perpendicular; and then multiplying one by the other, half the Product is the Area; or half one, Multiplied by the other, is the Area)

But in some certain Cases 'tis necessary to know how to Measure these things to such a Degree of Exactness as cannot be attained, but by means of Calculation: And therefore I shall here give you the Principles, on which all that Art depends.

28. *In a Rectangle Triangle* a b d, *two sides being given; to find the Third, by Calculation.*

Let the Leg *b d* be 3 Fathom, (*See the last Figure*) and the Leg *a d* 4 Fathom, multiply 3 by 3, and 4 by 4, and you will produce the two Squares, 9 and 16. The summ of which two Squares is equal to the Square of the Hypothenuse *a b*: (6. 61.) and consequently the Square of *a b* is 9 added to 16, *i.e.* 25. And therefore to find *a b* I need only extract the side or square Root of 25 which is 5: and I conclude *a b* to be 5 Fathom.

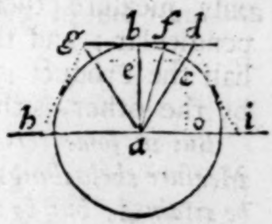
If the Hypothenuse *a b* 5 be given, with the side *a d* 4, you must then substract the Square 16 from the Square 25, there will remain 9, whose Square Root 3, is the Leg *d b* sought. But it often happens that the two Squares of the Legs added together, do not make a square Number: and also as often, the square of the one Leg substracted from the square of the Hypothenuse doth not leave a square Number: as if the Legs had been 2 and 3, their squares had been 4 and 9;

whose

whose summ is 13. But 13 is not a square number, and consequently can have no exact Root, nevertheless there are numbers which will come near it. As $3\frac{3}{5}$ is pretty near the Root of 13, for $3\frac{3}{5}$ multiplied by it self, makes 13 within $\frac{1}{25}$ part: so that the side *a b*, in this case, would be $3\frac{3}{5}$ and a little more.

I don't shew here the way of Extracting the square Root: because 'tis a Rule in Arithmetick, of which I dont Treat here.

29. *To Calculate the Tangent, Secant, and Sine of 30 degrees.* Let *b a* be the Radius or *whole Sine*, *a d* the Secant of 30 degrees, *b d* the Tangent, and *c e* the sine of the same same Ark. 'Tis easie to see that *b d*, here is the half of *a d*, For drawing *a g* another Secant of 30 degrees, the Triangle *a g d* will be equilateral because each Angle *g*, *d*, and *g a d* will be 60 deg. wherefore *b d* being half of *g d* must also be the hal *a d*. And for the same Reasons *c e* will be the half of *a c*. Supposing then in the Rectangle Triangle *a e c* the Hypothenuse *a c* be 2, and the side *e c* 1, therefore taking the square 1, from the square 4, there remains 3, which is equal to the square of the side *a e*, which is equal to *o c*, the sine of the Ark *c i*, or 60 degrees.



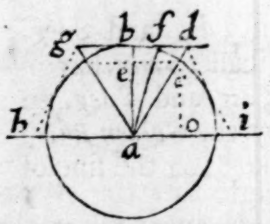
But now if instead of taking 2 and 1, for *a c* and *e c*, you take, 1000000 and 500000, and substract the later from the former, there will remain 750,000,000,000 whose square Root is very near 866025, for *a e* or *o c*, the sine of 60 degr.

30. *Hence knowing* e c *the Right-Sine of any Angle*, c o *the Sine Complement, or* Co-sine, *of that Angle is known.* The Complement of any Angle or Ark is what it wants of 90 deg. thus having the Angle *c a b*, 30 *deg.* its Compement *c a i*, 60 *deg.* is also known, by Substracting 30 from 90. And its Sine is found by the precedent.

31. *Knowing* e c *the Sine of any Angle and* c o *or* a e *its Sine Complement to find the Tangent* b d *and the Secant* a d. Since the Triangles *a e c* and *a b d* are Similar it follows,

that, $ae.ec::ab.bd.$ and consequently the Tangent is found; Also, as $ae.ac::ab.ad.$ and therefore the Secant is found, and the Ark cb being 30 *deg.* the Tangent bd will be 577, 350 and the Secant ad 1, 154, 700 such parts, as the Radius is supposed to be divided into.

32. *Knowing the Sine, Tangent and Secant of any Ark* bc, *to find the Sine, Tangent and Secant of half that Ark.* Bissecting the Ark bc, draw af from the middle Point to the Centre and then $df.fb::ad.ab.$ (6. 72.) and consequently bf the Tangent of 15 *deg.* is found, is also the Sine and Secant of the same Ark of 15 *deg.* After which by Bissecting the Ark bf, you may, by the same way, find the Tangent Sine and Secant of 7 *deg.* 30 *min.* and after that of 3 *deg.* 45 *min.* and so on as you please.



33. *To find* ce *the Sine of* 45 *deg.* This is equal to the Cosine of the same Ark of 45 *deg.* that is to ea and consequently the Tangent and Secant of that Ark is also found, and (by the last) of its half 22 *deg.* 30 *min.* and of the half of that 11 *deg.* 15. *min.* &c.

34. *To find the Sine of* 36 deg. Inscribe a Regular Pentagon in the Circle, the Proportion of whose side to the Radius is known (9. 13.) But that side is the Chord of 72 *deg.* and therefore its half will be the sine of half 72 *deg.* that is of 36 *deg.* And the sine of 36 *deg.* being thus known, the Tangent and Secant of the same Ark are known also: And also the Sine, Tangent and Secant of the half of 36 *deg.* which is the sine of 18 *deg.* And then of 9 *deg.* 4 *deg.* 30 *min.* 2 *deg.* 15 *min.* &c.

35. *To find the Sine, Tangent and Secant of* 12 deg. *and of the halves* 6 deg. 3 deg. 1 deg. 30 min. 45 min. *&c.* This is done by finding the Chord of 24 *deg.* which is the side of a regular Polygon of 15 sides (9. 13.) for its half is the sine of 12 *deg.* &c.

36. And Collecting all these together we shall have the Sines, Tangent and Secants of the Angles of 45 *min.* of 1 *deg.* 30 *min.* of 2 *deg.* 15 *min.* of 3 *deg.* 45 *min.* of 4 *deg.*

30

30 *min.* and ſo of all the reſt from 45 *min.* to 45 *min.*

37. *To find the Sines, Tangents and Secants of all Arks that are between the two Arks thus found from* 45 min. *to* 45 min. This is done by the Rule of Proportion: For inſtance, the natural ſine of 45 *min.* being 1308, the ſine of 1 *min.* will be 29. becauſe as 45 *min.* 1 *min.* : : 1308. 29. And thus the ſine of 20 *min.* will be 581.

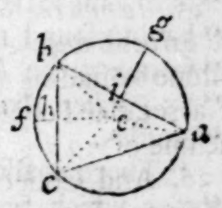
So alſo to gain the ſine of 3 *deg.* 30 *min.* you muſt Work thus; having the ſine of 3 *deg.* 5233. (9. 35.) as alſo the ſine of 3 *deg.* 45 *min.* which in 6540. (9. 32.) 'tis manifeſt that theſe 45 *min.* which are between 3 *deg.* and 3 *deg.* 45. have for their ſine 1307 for Subſtracting 5233 the ſine of 3 *deg.* from 6540 the ſine of 3 *deg.* 45 *min.* There remains 1307. If therefore you would have the ſine of 3 *deg.* 30 *min.* you muſt ſay,

If 45 *min.* (the difference between 3 *deg.* and 3 *deg.* 45) have for their ſine 1307; what ſhall 30 *min.* have? (which are the difference between 3 *dig.* and 3 *deg.* 30 *min.*) the anſwer will be (working by the *Golden Rule*) 873. which if added to 5233 will make 6104 the ſine of 3 *deg.* 30 *min.* And ſo of all others.

And by this means may Tables of (Natural) *Sines, Tangents and Secants be made, from one Minute to* 90 deg.

Obſerve that by the laſt Rule the ſines cannot be found exactly, becauſe the ſines do not encreaſe in the ſame Proportion as their Arks do; but the difference is ſo very ſmall that you need not Trouble your ſelf to be more exact.

38. By the help of ſuch Tables as theſe, Triangles may be Calculated; becauſe 'tis certain that in all Triangles, the ſides are to one another as the ſines of the oppoſite Angles. For Example, In the Triangle *a b c*, a Circle being drawn which ſhall Circumſcribe it, the Perpendiculars *e i* and *e h* drawn from the Centre to the ſides ſhall Biſſect *a b* and *b c* (4. 6.) ſo that *a b. b c* : : *a i. b h.* But *a i* is the ſine of the Angle *a e i* or its equal *a c b* (4. 13.) and *b h* alſo is the ſine of the Angle *f e b* or of *b a c* which



is

is equal to it. Wherefore the ſides are as the ſines of the oppoſite Angles. Q. E. D.

39. And upon this Principle, *Knowing one Angle and two Sides, or one Side and two Angles you may find the reſt.* Work by the *Golden Rule* and ſay, as one ſide known is to the ſine of the Oppoſite known Angle; : So is the other known ſide, to a fourth Number: which will be the Angle oppoſite to that other ſide.

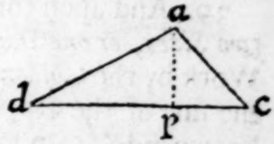
Or if two Angles and one ſide are given, you muſt Work thus, as the ſine of one given Angle, is to the known ſide oppoſite to it: : So is the ſide of the other known Angle to a fourth Term which will be the ſide oppoſite to that other Angle, *&c.*

"(N. B. By this Axiom, as 'tis called in Trigonome-"try, *that the ſines are as the ſines of the oppoſite An-"gles*; you can only ſolve the three firſt Caſes of Ob-"lique Triangles; which our Author ſhould have Inti-"mated. For when *two Sides* and the *Angle between* "*them* in an oblique Triangle are given, and either the "*Oppoſite Side*, or the *other two Angles* are requir'd: You "muſt Work by this Axiom, and firſt find the other An-"gles thus: As the Sum of the Legs about the Angle "given, is to their Difference : : So is the Tangent of "half the Sum of the other two Angles, to the Tangent "of half their difference. Now the Sum of the other "two Angles is known, being what the given Angle "wants of 180 *deg.* and their difference is now found, "add therefore their half Sum, and half difference to-"gether and it gives you the *greater of the Two Angles* "*ſought*; and half the difference Subſtracted from the "half Sum leaves the leſſer Angle ſought. And thus "having found the Angles; if the ſide oppoſite to the "former given Angle be ſought, it will be found eaſily, "by *Pardies Axiom* that the ſides are as the ſines of the "Angles.

"There is alſo another Caſe in plain oblique Trian-"gles which requires a particular Axiom to ſolve it; "and that is, where *All three ſides are given to find the* "*Angles.* Here, let fall a Perpendicular from any An-

"gle

“ gle to its opposite side as *ap*. And then say, *as the side* “ d c *is to* d a + a c *the Sum of the* “ *other two Sides* :: *So is the diffe-* “ *rence of those two Sides* d a — “ a c *to a fourth Number*. Half “ of which added to $\frac{1}{2}$ *dc*, gives “ you the Segment of the Base



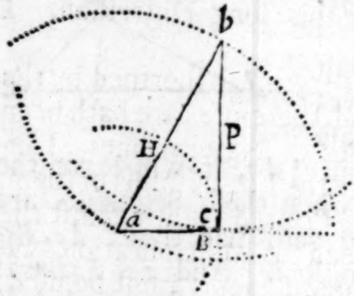
“ *dp*; and if Substracted from $\frac{1}{2}$ *dc*, it will leave the “ other Segment *pc*. And when those Segments are “ thus found, the Angles are easily had thus: *da*. *Ra-* “ *dius* :: *dp*. Co-sine of the Angle *d*. And *ac*. *Radius* :: “ *pc*. Co-sine of the Angle *C*. See a very short and clear “ Demonstration of these two Axioms in Mr. *Caswells* “ Trigonometry *p*. 5.

“ And thus you have here a short Compendium of all “ plain Trigonometry, only *Pardie* hath not particularly “ insisted on the solution of Right-angled Triangles a- “ part; But tho' it be true that these Axioms here men- “ tioned will help us to solve all plain Triangles what- “ soever, whether Right or Oblique; yet 'tis more con- “ venient to consider Rectangled ones, as a distinct “ Species by themselves, as is usually done in all Trigo- “ nometrical Books: For in them the Right-angle being “ always known, you need have but two Terms given “ you, to find the rest. And for the Solution of all the “ seven Cases of Rectangled Triangles you need only “ Consider, that which side soever you make or suppose to “ be the Radius, the other two will be either Sines, Tan- “ gents or Secants. If the Hypothenuse be made Ra- “ dius, the Legs will be the sines of the opposite Angles; “ if either of the Legs be made Radius, the Hypothe- “ nuse will be the Secant, and the other Leg the Tangent “ of the Adjacent Angle, or the Angle at the Centre of “ the supposed Circle, as is plain from the following “ Figure.

“ And therefore in order, to solve any of the Cases of “ Rectangle Triangles (or indeed of Oblique ones) you “ are to imagine one Triangle before you given to be “ resolved, and another Equal and Similar to it, sup-

“ posed

"posed to be in the Tables of Logarithms, Sines, Tan-
"gents, &c. and therefore the sides of these two Triangles about the equal Angles will always be Proportional.

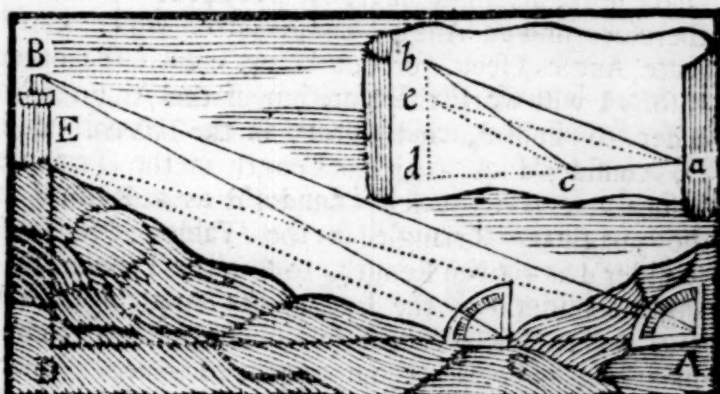


"This therefore (which "is the ground of all Trigonometry) being suppos'd, you will easily solve "any of the Cases of Right-angled Triangles: Thus "Let the Hypothenuse H, "and B the Base be given, and P the Perpendicular be "required. Here in the first place I consider that a "Length is required, as two Lengths are given; but "I have never an Angle but the Right one c. I must first "therefore find an Angle, 'tis no matter which of the "acute Angles I seek, for one being known the other "is so. I will work therefore about the Angle a, in "order to find b, and I say as the Hypotheause "H, consider'd as a given Length in the Triangle "before me, is to it self consider'd as a Radius in "the imaginary Triangle in the Tables :: So is B "consider'd as a given Length, to B in the Tabular Triangle consider'd as the Sine of the Angle b. That "is $H. R :: B. S, b$. Wherefore b is known; and consequently a, as being its complement to 90 *deg*. Then "knowing a I know its Tangent, from the Tables. I "will make therefore B the Radius and Work my Proportion about the Right-angle thus; I say, as B considered as a Radius is to it self considered as a Length: "So is P consider'd as the Tangent of the Angle a, to "it self consider'd as a Length, at first sought or required. That is $R. B :: T, a. P$. And therefore the Length "of the Perpendicular P will be known. This fourth "Case of Right-angled Triangles I chose out from "among the rest because tis the most difficult of any of "them; and therefore this being explained, as I hope "it is here, there will be no difficulty in solving any of

"the

"the other Cases. By this Instance and Explication al-
"so you may get a rational and clear Notion of all Tri-
"gonometrical Calculation: Wherefore I doubt not
" of the Readers Pardon for this long Digression. I
" returnto our Author.

40. These Operations are very much shortned by the Use of the Logarithims; And therefore care hath been taken not only to make Tables of Natural Sines, Tangents, &c. but also to Calculate the Logarithims of them, which you will find in Books Printed right against the Natural ones. So that instead of Multiplication and Division, which, with intolerable Labour, must be used, if you Calculate by the Natural Sines and Tangents; you need now only use Addition and Substraction when you work by the Logarithims. As if in the Triangle *ABC*



whose Side *AC* is known, and is 10 Fathom; the Angle *ABC* is given and is 10 *deg.* and the Angle *CAB*, of 20 *deg.* is given also, the Side *BC* be required: You must say as the Sine of the Angle *B*, (which in the Natural Sines is 17364) is to the Side *AC* (10 Fathom); so is the Sine of the Angle *A* (which in the Tables is 34202) to the Side sought, *CB*. To find which fourth Number *CB* by the *Rule of Three*, you must multiply the second Term 10, by the third Term 34202, and divide the Product 342020, by the first Term 17364 which would be very tedious. But if instead of these

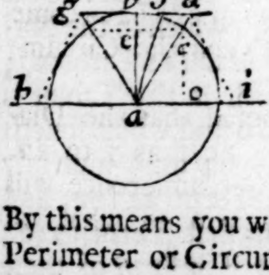
Natura

Natural Numbers you had used the Logarithms, you need only have added the Logarithm of 20 *deg.* to the Logarithm of 10 Fathom and from their Sum substracted the Logarithm of 10 *deg.* there would have remained the Logarithm 1.2943815 which in the Tables will be found to belong to the absoluthe Number 19; so that you may conclude the side *BC* is 19 Fathom and something more because the Logarithm found is greater than the proper Logarithm of 19 in the Tables. See the Work above.

Sin. Ang. *A* 20 *deg.*	9.5340517
Logarith. *AC.*	1.0000000
Their Sum.	10.5340517
The Sin. An. *B* 10 *deg.*	9.2396702
There Remains	
That is 19$\frac{7}{10}$ Fathom.	1.2943815

Books that Treat of Sines and Logarithms do explain this matter more particularly and fully; but I suppose I have said enough here to make any one able to understand Calculation, without the help of a Master: I shall also in the subsequent Books of Geometry give you some more Propositions on this Subject.

41. *To find a Right-line that shall be very nearly equal to the Circumference of a Circle.* Taking the Tangent of 30 *deg.* which is *b d* twelve times, and disposing these Tangents so round about the Circle as that they be joined (in a Right-line) two and two together as you see in the Figure: Where *d g* is composed of two Tangents so joined each of 30 *min.* and so is *g h* and *d i*, &c.



By this means you will make a Polygon of six Sides whose Perimeter or Circumference is greater than that of the Circle

Circle (4. 27.) And if you take the Sine *ce* twelve times, proceeding as before with the Tangents, you will have an inscribed Polygon of six Sides whose Circumference is less than that of the Circle. So that taking the Radius *ab* 1, 000, 000; *bd* which 11, 577, 350, taken twelve times, that is to say, 6, 928200 is greater than the Circumference of the Circle. And *ec* which is 5000,000, taken twelve times, that is, 6000000, is less than the Circumference of the Circle.

42. But if instead of taking twelve times the Tangent and Sine of 30 *deg*. you take 360 times the Tangent and Sine of one Degree: That is 17455 and 17452 you will make two Polygons, the one Circumscribed which will be 6283800 and greater than the Circle, the other Inscribed 6282720 which will be less than the Circle.

43. Again, let the Radius 10000000000 be given, taking the Tangent and Sine of one Minute 216000 times (which is the Number of Minutes in 360 *deg*.) you will have 628318532600 a little greater and 6283185120000 a little less than the Circle for the Tangent of one Minute is 29088821, and the Sine of one Minute is 29088820. And if these three Numbers the Radius, the Circumscrib'd Polygon and the Inscrib'd Polygon, be divided by 1000000: There will remain for the Radius 1000000: And the Perimeter of the Circumscrib'd Polygon will be 6,283185$\frac{326}{1000}$ and the Perimeter of the Inscribed Polygon will be 6283185$\frac{12}{100}$. So that these two Perimeters, the one greater and the other less than the Circle, will differ but by one hundred thousandth Part of the Radius. And if you had taken the Tangent and Sine of one second, you might have come yet vastly nearer to an Equality between the Circumscribed and Inscribed Polygons.

44. For Practice, 'tis usually supposed that the Diameter is to the Circumference very near as 7 to 22. That is, if the Radius be 7, the Circumference will nearly be 44 such Parts, and this agrees to what we have here deliver'd; for 7. 44 :: 100. 628$\frac{4}{7}$.

45. *To*

45. *To find the Area of the Circle.* If the Radius be supposed 1000, the Circumference will be very nearly 6283: half of which, *viz.* 3131 being multiplied by the Radius 1000, Produces 3141000 for the Area of the Circle (4. 31.) But if the Radius be taken in any other Measure, as suppose 9 Inches; you must say as 1000. 3141 :: 9. 16 $\frac{269}{1000}$. And then Multiplying this last Number 16 $\frac{269}{1000}$ which is the Semicircumference by 9, the Radius, you will have 173 $\frac{427}{1000}$ for the Area of the Circle to that Measure.

And in my Judgment 'tis better to use this Proportion of 1000 *to* 3141 *than the Common one of* 7 *to* 22.

46. *To find the Solid Content of a Parallelepiped or of a Cylinder.* Multiply the Base by the Perpendicular Altitude.

47. *To find the Content of a Pyramid or Cone.* Multiply the Base by one third of the Perpendicular height.

48. *To find the Content of a Sphere.* Multiply the Surface of one third of the Radius; or a great Circle by $\frac{2}{3}$ of the Diameter.

FINIS.

A
TABLE
Of the Words and Terms of Arts explained in this Geometry.

N. B. The First Figure shews the Book, the Second the Article.

A.

Base

B.

C.

D.

Diagonal

Part

P.

Q.

R.

Triplicate

The End of the TABLE.

The Characters, Marks, Signs or Symbols here used, are only these,

= Equal to.
+ More or adding.
− Less or Substracting.
:: The Mark of four Quantities being discretely Proportional Geometrically.
∺ The Mark for Continual Proportion, or Geometrick Progression.
::: The Mark for Arithmetical Proportion.